A Signals and Systems Primer

(for avionics and communication systems)

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These handouts aim to provide a fast-track introduction to elementary concepts in signals and systems that underpin the design and implementation of avionics and communication systems. These concepts arise in such diverse applications in aeronautics and astronautics, communications, acoustics, seismology, speech and image processing and control systems.

Signals are functions of time that convey information. For example, sensor readings over time constitute a signal. A microphone is a sensor that converts acoustic pressure to an electrical signal which is sometimes called an audio signal.

Systems transform signals to other signals that are more convenient in the context of a selected application. For example, a signal might be noisy and a de-noising system would remove the noise and transform the noisy signal to a signal of higher quality. In the context of sensor readings, this would remove the nuisance volatility in the measurements, or in the context of audio communication, this would improve the hearing quality.

Aviation electronics and communication systems are systems that extract information necessary for the use of an aircraft and facilitate the exchange thereof. Here, information might be relevant to navigation, surveillance, air traffic management, or other aspects of aviation.

Signals and systems refers to mathematical analysis tools that are used by engineers to specify, analyze and design systems including communication and avionic systems. These notes introduce fundamental elements for the investigation of signals and systems.

Contents

This document introduces signals, their representations, systems as entities that manipulate signals and their characterisation. As such, we start with a review of mathematical fundamentals to provide a smooth transition to the discussion of signals and systems as extensions of these fundamental notions.
Section 1 introduces sets of numbers and functions. Trigonometric functions and complex numbers are extensively used in the study of avionics and communication systems which are explained in Section 2.

The part on signals start by defining signals as functions of time in Section 3. Then, periodic functions that repeat a pattern over time are introduced as a basic signal class in Section 4. Fourier series expansions of such signals are given to represent them using well-known functions as building blocks.

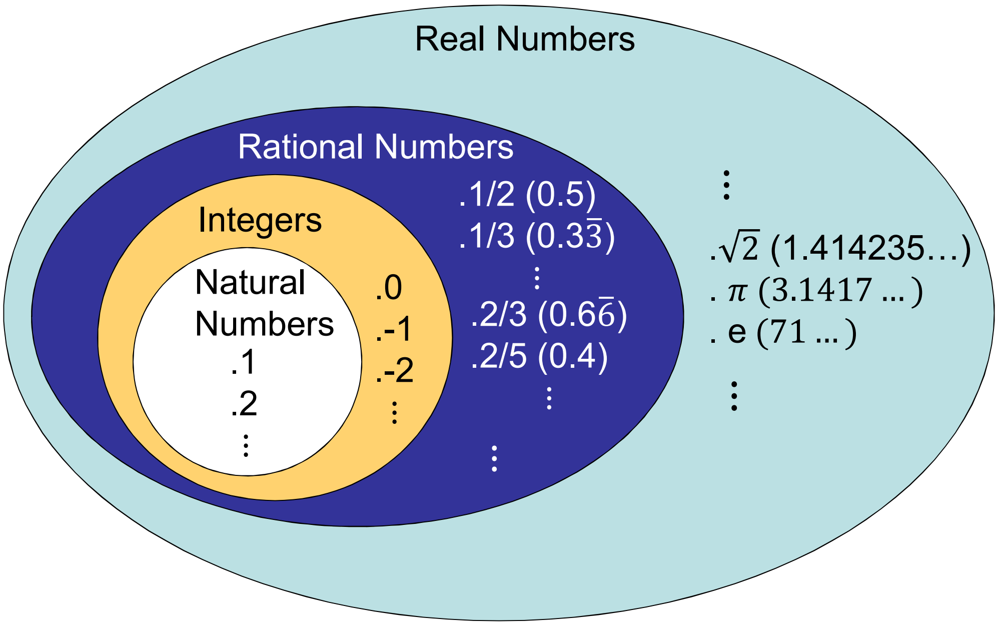
Section 5 defines non-periodic signals as the complementary class of signals and introduces The Fourier Transform as a key tool to reveal their contents in terms of well-known functions.

The part of systems starts with the definition of systems in Section 6. A particularly useful class of systems are the linear and time-invariant (LTI) systems. This class is introduced in Section 7 and their characterisation in Section 8. Section 9 explains how to find the output of an LTI system to any input based on how they manipulate well-known functions.

Finally, we provide a summary in Section 10.

The above provides an introduction to Signals & Systems, which consists of further topics in dedicated modules in engineering curricula. Topics beyond an introduction are left out of the scope of this module, e.g. explicit Fourier Series/Transforms computations, LTI filter design, z-transform, the discrete-time Fourier Transform, discrete Fourier Transform and the Fast Fourier Transform algorithm are left out of the scope of this document.

1.A Numbers



First, let us consider the set of natural numbers, or counting numbers, often denoted by $\mathbb{N} = \{1,2,3,4,5, \dots\}$. This set has infinitely many elements but it is countable.

The set of integers $\mathbb{Z} = \{\dots, -5, -4, -3, -2, -1,0,1,2,3,4,5, \dots\}$ extend $\mathbb{N}$ with negative numbers and one of the most important number of all: zero.

Rational numbers $\mathbb{Q}$ are numbers that can be written as the ratio of two integers excluding division by zero. All decimals that terminate (e.g. 3.12322 as it equals to 313222/100000, which is a ratio of two integers) and all repeating decimals (e.g. $0.6\bar{6}$) are rational numbers.

Real numbers $\mathbb{R}$ extend rational numbers to include those which cannot be written as a ratio of two integers. For example, $\sqrt{2}$ is an irrational number which is the diagonal length of the unit square with edge lengths of one. Note that in decimal form, $\sqrt{2}$ is non-terminating and non-repeating. Another important irrational number is the circumference to diameter ratio of the unit circle, π $(3.1417 \dots)$. The list of important irrational numbers is never complete without Euler's number e $(71 \dots)$, which is the base of the natural logarithm.

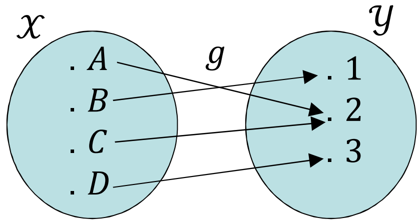
One important property of the set of real numbers $\mathbb{R}$ is that it is "uncountable", i.e. it is not possible to index its elements as one could the elements of, for example, $\mathbb{Z}$. There are infinitely many real numbers and they can be ordered to form a "real line"; zero placed at the centre, the line will have numbers increasing in magnitude to the right and negative numbers increasing in magnitude to the left. There are infinitely many real numbers in any non-zero length interval of the real-line.

These sets are endowed with the operations of summation, multiplication and their inverses subtraction and division to model and solve many real-world problems that involve "quantities" such as finding distances, areas and volumes since ancient times.

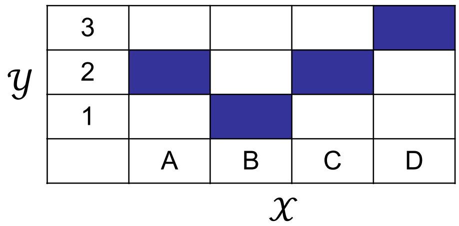
1.B.1 Functions

- A *function* maps elements of its *domain* set to elements in its *range* set, such that only one element of the range is assigned to a selected element of the domain.

- On the right-hand side is a function $g: \mathcal{X} \rightarrow \mathcal{Y}$ whose domain is $\mathcal{X} = \{A, B, C, D\}$, and range is $\mathcal{Y} = \{1,2,3\}$.
- The function f can also be represented by listing *ordered pairs*: $g = \{(A, 2), (B, 1), (C, 2), (D, 3)\}$.



- g can also be represented by *a table* that marks the ordered pairs.



A function (or mapping) from a set X to a set Y is a rule that assigns an element of X to one and only one element of Y. The set X is called the domain of the function, and the set Y is called the range (or codomain) of the function. This is often denoted by $g: X \rightarrow Y$ (and read as g sends/maps elements in X to elements in Y).

When X and Y are finite, g can be represented by listing the assignments it specifies as ordered pairs such that the element in the domain is written first, and the assigned element in the range is written second. On the slide is an example $g: \mathcal{X} \rightarrow \mathcal{Y}$ for which, for example, (A,2) is an ordered pair with $\mathrm{A} \in \mathcal{X}$ and $2 \in \mathcal{Y}$.

Note that a function cannot assign two elements of its range to the same element of its domain. For example, if the list of ordered pairs contained (A,3) in addition to the existing pairs, then g would not be a function because $\mathrm{A} \in \mathcal{X}$ would have two values in g. Then, g would be referred to as a “relation.”

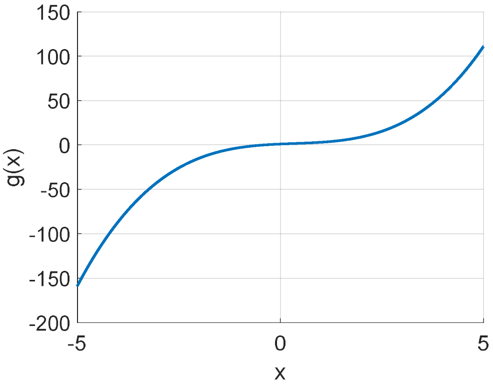
A tabular representation of a function g can be obtained by listing the elements of its domain along the columns and its range along the rows of a table, and then marking the ordered pairs that specify the function g.

Functions can be classified according to some of their properties/attributes; for example, if there is no element in $\mathcal{Y}$ that is mapped to more than one elements in $\mathcal{X}$, g is called a one-to-one function.

Some more details on the topic can be found in https://en.wikipedia.org/wiki/Function_(mathematics)#Injective,_surjective_and_bijective_functions

For a discussion on signals and systems, let us consider real functions next.

1.B.2 Functions



- A real function maps real numbers to real numbers, i.e. $g: \mathbb{R} \rightarrow \mathbb{R}$. For example polynomials are real functions.
- Let us consider the polynomial function given by ordered pairs
  $$g = \{\, (x, x^3 - x^2 + 2x + 1)\ such\ that\ x \in \mathbb{R}\}.$$
- g can also be represented by depicting its *graph*, i.e. the locus of the ordered pairs it specifies similar to the *table* in the previous example.
- A more common representation for real functions is to use their general rule as $g: x \rightarrow x^3 - x^2 + 2x + 1$, or shortly,
  "$g(x) = x^3 - x^2 + 2x + 1\ \forall\, x \in \mathbb{R}$".
- Composite functions chain more than one function for their mapping.
- For example $g \circ h: \mathbb{R} \rightarrow \mathbb{R}$ denotes a composite function of two functions $g: \mathbb{R} \rightarrow \mathcal{Y}$ and $h: \mathcal{Y} \rightarrow \mathbb{R}$ with the general rule $\mathrm{g} \circ h(x) = g(h(x))$.
- Here, $\mathcal{Y}$ can be a set different than $\mathbb{R}$ provided that g and h satisfy the requirements of a function.

Functions can have infinite domain and range sets such as the integers or real numbers.

A function $g: \mathbb{R} \rightarrow \mathbb{R}$ (in other words, g that maps real numbers to real numbers) with the set of real numbers as both its domain and range is referred to as a real function. Such functions are sometimes called real-valued functions. Because the ordered list representation of a real function would be an infinite list, it would not serve well to our purposes. One common way of representing a real functions is to write an ordered pair with its general rule. For example, polynomials are real functions with general rules that take the powers of elements in their domain sets, multiply them with some coefficients and finally sum all of them together.

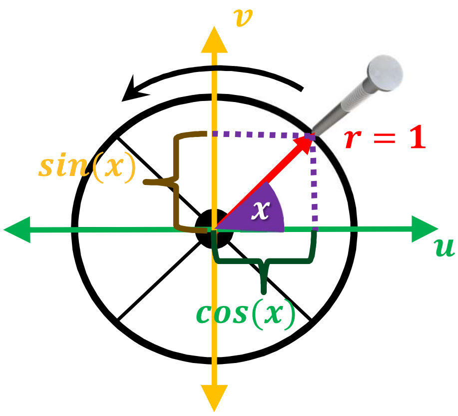
In the example above, a real number x is mapped to a summation of its 0th , 1st, 2nd and 3rd power after multiplying them with the coefficients of +1, +2, -1 and +1, respectively.

A more common representation is to use the general rule of a function as in "$g(x) = x^3 - x^2 + 2x + 1$ (g of x equals to x cubed minus x squared plus two x plus one) $\forall x \in \mathbb{R}$ (for all x in the set of real numbers)".

A visual representation of real functions are via their graphs. The graph of a real function is similar to the tabular function representation in the previous slide: the vertical axis is the domain and the horizontal axis is the range of the function. As real numbers can be ordered to form a continuum, we use the resulting real line on both axes. Then, the pairs $(x, g(x))$ are marked on this "table" for $x \in \mathbb{R}$ to depict the graph of the function g. Note that the real line axes have both negative and positive sides, and the number zero.

Composite functions are built by cascading a number of functions: For example, two function $g: \mathbb{R} \rightarrow \mathcal{Y}$ and $h: \mathcal{Y} \rightarrow \mathbb{R}$ can be cascaded to obtain a composite real-function $g \circ h: \mathbb{R} \rightarrow \mathbb{R}$ for which the general rule becomes $g \circ h(x) = g(h(x))$. Here, $\mathcal{Y}$ is the range set of g and the domain set of h. It can be that $\mathcal{Y} = \mathbb{R}$, or $\mathcal{Y}$ can be any other set. The crucial point is, g and h should satisfy the requirements of a function, i.e. map an element of their domain to one and only one element in their range.

2.A.1 Sinusoidal Functions



- Consider a spinning wheel of radius $r = 1$ with a nail attached to mark a specific point.
- For a frozen time instant, x is the angle between the horizontal axis and the line segment conjoining the centre of the wheel to the marked point (in radians).
- Positive x indicates a counter clock-wise rotation and a negative x is a clock-wise rotation.
- The functions that give the horizontal and vertical placement of the nail from the centre are called the **cosine** and the **sine**, respectively.

- The horizontal coordinate of the nail along the $\boldsymbol{u}$ axis is given by $cos: \mathbb{R} \rightarrow$ [-1,1] with the general rule denoted by $u = \boldsymbol{cos(x)}$.
- The vertical coordinate of the nail along the $\boldsymbol{v}$ axis is given by $sin: \mathbb{R} \rightarrow$ [-1,1] with the general rule denoted by $v = \boldsymbol{sin(x)}$.
- The pair $(\sin(x), \cos(x))$ is called the coordinates of the (nailed) point on the $\boldsymbol{u} - \boldsymbol{v}$ plane.
- Note that $(\cos(x))^2 + (\sin(x))^2 = 1$.

Trigonometric functions constitute an important function class for reasons that will become clear later.

Let us start by considering a spinning wheel of radius $r = 1$, and a certain point on the wheel which is marked with a nail. We are interested in the horizontal and vertical location of the this point at a frozen moment.

Let us denote by x the angle from the horizontal axis to the nail. The unit of x is radians, or rad for short: 0 rad aligns with the horizontal axis $\boldsymbol{u}$, $\pi/2$ rad aligns the vertical axis, 2π rad corresponds to one full cycle in the counter clockwise direction and aligns back with the horizontal axis. When x equals to integer multiples of 2π rad, it corresponds to the nail completing the integer multiple many times full rotation. For example, if $x = 10 \times 2\pi$, the line segment conjoining the centre must have rotated 10 times starting from a perfect alignment with the positive $\boldsymbol{u}$ axis and arrive back at perfect alignment. What would $\mathrm{x} = 2\pi + \frac{\pi}{3} = \frac{7\pi}{3}$ rad correspond to?

The function that map any real value of x to the horizontal displacement of the nail from the vertical axis is called the cosine function. Because the radius of the wheel is unity (i.e. one), the values that cosine produces are between -1 and 1, inclusive. These values constitute an interval shown by [-1,1]. Thus, $cos: \mathbb{R} \rightarrow$ [-1,1].

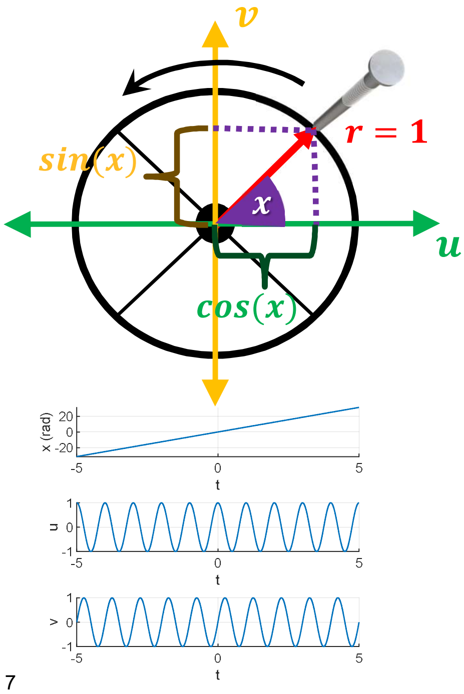
Similarly, the function that map any real value of x to the vertical displacement of the nail from the horizontal axis is called the sine function, and $sin: \mathbb{R} \rightarrow$ [-1,1].

The pair $(\sin(x), \cos(x))$ unambiguously locates the nailed point on the $\boldsymbol{u} - \boldsymbol{v}$ plane and is called the coordinates of the point. Because this point is on the circle with unit radius, the distance of this point from the origin is one. This is also verified by the Pythagorean Theorem that states the sum of the squares of the coordinates equals to the square of the radius here which is one.

How can you recover x if given only the values of $(\sin(x), \cos(x))$? The function that maps (sin(x), cos(x)) values to x is called the arctangent function (see, https://en.wikipedia.org/wiki/Atan2).

In practice, we can calculate both cosine and the sine very accurately for any value of x. Most programming languages and interpreters such as MATLAB provide built-in functions to evaluate cosine and sine.

2.A.2 Sinusoidal Functions



- Suppose the wheel is rotating with a constant rate of increase in x.
- This is described using a function $g: \mathbb{R} \rightarrow \mathbb{R}$ that outputs x for any given time instant $t \in \mathbb{R}$, i.e. $x = g(t)$ with the general rule, e.g. $g(t) = 2\pi t$.
- Equivalently $x = 2\pi t$.
- The composite function $cos \circ g: \mathbb{R} \rightarrow [-1,1]$ thus has the general rule $u = \cos(2\pi t)$.
- Similarly $sin \circ g: \mathbb{R} \rightarrow [-1,1]$ has the general rule $v = \sin(2\pi t)$.
- The graphs of these functions for $t \in [-5,5]$ are on the left.
- Note that the graphs of u and v are repetitive (or periodic).
- Also note that
  - $\cos(-2\pi t) = \cos(2\pi t)$
  - $\sin(-2\pi t) = -\sin(2\pi t)$
- Both functions appear as shifted versions of each other. Why?

Now suppose that the wheel has been in perpetual rotation. This can be described by using a function $x = g(t)$ that maps a given time instant $t \in \mathbb{R}$ to an angle value x. We further consider rotation with a constant rate of increase in x. Linear functions have constant rate of increase, and an implicit assumption here is alignment of the red vector with the positive direction of the horizontal axis at $t = 0$ (i.e. our time origin reference). Therefore, $g(t) = 2\pi t$ is an example function which can describe such rotation, i.e. $x = 2\pi t$; the tip of the nail is at +1 on the u axis at time zero, and it will have completed one rotation at each second. Can you find other $x = g(t)$ that satisfy the above requirements of i) constant rate of increase and ii) alignment with the positive direction of the horizontal axis at $t = 0$?

The composite functions of cosine and sine with g thus give the coordinates of the marked point during this perpetual rotation. In other words, the pair $(u, v) = (\cos(2\pi t), \sin(2\pi t))$ tells us the exact location of the nailed point on the $\boldsymbol{u} - \boldsymbol{v}$ plane at any time $t \in \mathbb{R}$. Note that both cosine and sine evaluate the same when integer multiples of $2\pi t$ is added to their arguments, i.e.

$\cos(2\pi t) = \cos(2\pi t + 2\pi) = \cos(2\pi t + 4\pi) = \cdots = \cos(2\pi t + m \times 2\pi) = \cdots$ for integer m, and similarly

$\sin(2\pi t) = \sin(2\pi t + 2\pi) = \sin(2\pi t + 4\pi) = \cdots = \sin(2\pi t + m \times 2\pi) = \cdots$

Given on the slide are graphs of these composite functions for $t \in$[-5,5]. Note that the maximum value of these functions is 1 and the minimum value they take is -1. Both of the sinusoidal functions are repeating themselves over time which is expected following the above property.

Here, a negative angle, i.e. $x < 0$ means a clock-wise rotation of $|x|$ radians starting from an alignment with the positive $\boldsymbol{u}$ direction. Keeping this in mind, one can see that the cosine is symmetric around zero as shown by $\cos(-x) = \cos(x)$; functions with this property are called even functions. The sine, on the other hand, is anti-symmetric around zero, i.e. $\sin(-x) = -\sin(x)$; functions with this property are called odd functions. They are also shifted versions of each other since as $\cos(x) = \sin(x + \frac{\pi}{2})$.

2.A.3 Sinusoidal Functions

- Let us consider the following case in which
  1) The wheel completes one full rotation in T seconds.
  2) The point marked with the nail is not at +1 on the $\boldsymbol{u}$ axis at time $t = 0$, but it makes an angle of ϕ with the $\boldsymbol{u}$ axis.
- This is described using the function

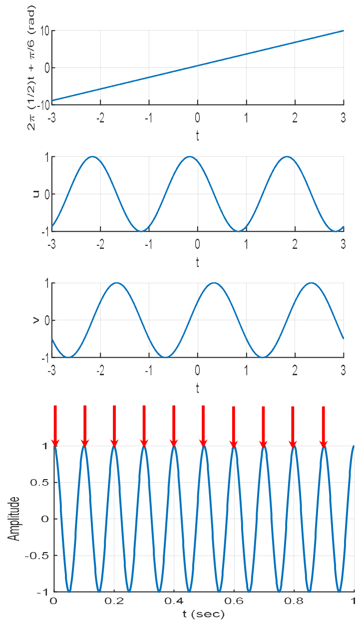
$$g(t) = 2\pi \frac{1}{T} t + \phi$$

- The rate of increase in angle is determined by $\frac{1}{T}$.
- The function $cos \circ g: \mathbb{R} \to [-1,1]$ now has the general rule $u = \cos(2\pi \frac{1}{T} t + \phi)$.
- Similarly $sin \circ g: \mathbb{R} \to [-1,1]$ has the general rule $v = \sin(2\pi \frac{1}{T} t + \phi)$.
- Graphs for $T = 2$ and $\phi = \frac{\pi}{6}$ are on the top right.
- $\frac{1}{T}$ is called the **frequency** of both u and v:

$$f = \frac{1}{T},$$

which also equals to the number of cycles in one seconds.

Examples:



$frequency = 10$ Hz

So far, we have not explicitly specified how the wheel's rotation was. Suppose that the wheel completes on full rotation in T seconds. This parameter provides flexibility in modelling slower or faster turns. A second parameter that we might want to vary is the angle between the radius towards the marked point and the $\boldsymbol{u}$ axis at time $t = 0$; we might want to model when this angle is non-zero unlike the previous discussion and takes the value of ϕ (phi) radians.

This function $g(t) = 2\pi \left(\frac{1}{T}\right) t + \phi$ captures these as i) it will have increased by 2π for every T increase in t, and ii) it maps $t = 0$ to ϕ. Consequently, the u and v values repeat themselves in every T second and evaluate at $\cos(\phi)$ and $\sin(\phi)$, respectively, at $t = 0$. The figures on the top right first give the graph of $g(t)$ vs time for evaluates $T = 2$ and $\phi = \pi/6$. Compare this graph to the graph of the angle function in the previous slide. Note that the rate of increase is smaller, here. Why? What is the period T in the previous example?

The u and v functions against time are given next. Note that because ϕ is non-zero, the cosine function now does not peak at $t = 0$ and the sine function does not cross zero $t = 0$. In fact, ϕ induces a shift to their graphs.

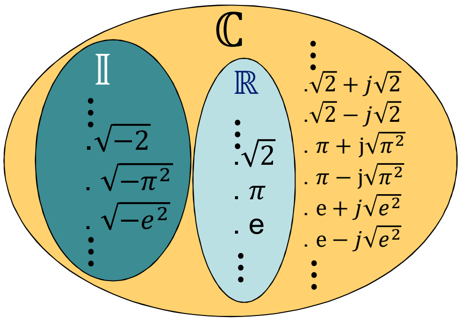
For the sinusoidal functions, $\mathrm{f} = 1/T$ is called their frequency. This quantity is a positive real number since $T > 0$ and $T \in \mathbb{R}$, in general, and equals to the number of cycles sinusoidal functions exhibit in one second. For further reading on these functions, see https://en.wikipedia.org/wiki/Sine_and_cosine

2.B.1 Complex Numbers

- All the elements of $\mathbb{R}$ result with non-negative values when squared: If $x \in \mathbb{R}$, then $x^2 \geq 0$.
- Numbers which have negative squares are called **imaginary numbers** and constitute the set of imaginary numbers $\mathbb{I}$.
  E.g. Both $\sqrt{-1} \times 2$ and $\sqrt{-1} \times (-2) \in \mathbb{I}$.
- In general, any imaginary number $\mathrm{a} \in \mathbb{I}$ can be written as the multiplication of $\sqrt{-1}$ with a real number, i.e. $\mathrm{a} = \sqrt{-1}\, b$ where $b \in \mathbb{R}$.
- The imaginary unit $\sqrt{-1}$ is often denoted by "i" in mathematics and "j" in engineering; e.g. $\mathrm{a} = j\, b$.
- A **complex number** is a sum of a real number and an imaginary number.
  The set of complex numbers is denoted by $\mathbb{C}$. Any complex number $\mathrm{z} \in \mathbb{C}$ is in the form

$$\mathrm{z} = z_{real} + j z_{imaginary},$$

where z_{real} and $z_{imaginary}$ are both real numbers. They are called the real and the imaginary part of the complex number z, respectively.



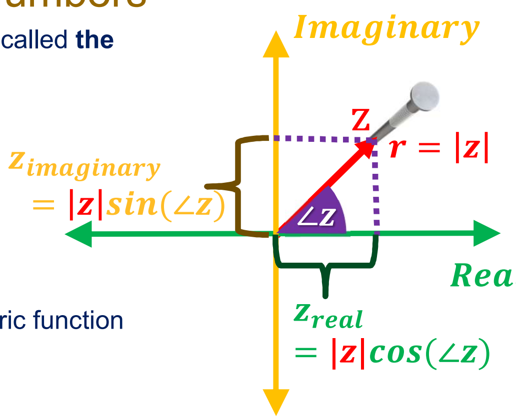
Trigonometric functions and complex numbers are related in many different ways.

Complex numbers arose from the solutions of polynomial equations. For example, consider the simple equation $x^2 = a$ for $\mathrm{a} \in \mathbb{R}$. If $\mathrm{a} > 0$, then there are two solutions which are $x = +\sqrt{a}$ and $x = -\sqrt{a}$. If $\mathrm{a} < 0$, is there a solution to this equation? Let us unpack this question: If the solution is sought in the set of real numbers $\mathbb{R}$, then there are no solutions, i.e. no element in $\mathbb{R}$ yields a negative number when squared. If we are allowed to use numbers outside of the set of real numbers $\mathbb{R}$, solutions do exist!

Numbers that have negative squares are called imaginary numbers. The set of imaginary numbers is often denoted by $\mathbb{I}$. For example, $2\sqrt{-1}$ is an imaginary number which yields -4 when squared. Similarly, $-2\sqrt{-1}$ squared is -4. Therefore, they are solutions to the equation $x^2 = a$ for $a = -4$. In general, the solutions of this equation for any negative a will have the form $x = +\sqrt{-1}\sqrt{-a}$ and $x = -\sqrt{-1}\sqrt{-a}$ and are imaginary numbers. Therefore, any imaginary number can be written as the multiplication of $\sqrt{-1}$ with a real number. In other words, if $\mathrm{a} \in \mathbb{I}$, then $\mathrm{a} = \sqrt{-1}\, b$ where $b \in \mathbb{R}$. Here, $\sqrt{-1}$ is referred to as the imaginary unit and denoted by the letter “i” in mathematics, and the letter “j” in engineering (since “i” is often reserved to denote electrical current). Thus, $\mathrm{a} = j\, b$.

A complex number is a sum of a real number and an imaginary number. The set of complex numbers is denoted by $\mathbb{C}$ and thus any element of this set $\mathrm{z} \in \mathbb{C}$ has the form $\mathrm{z} = z_{real} + j z_{imaginary}$ where z_{real} and $z_{imaginary} \in \mathbb{R}$ and called the real and the imaginary part of z, respectively.

2.B.2 Complex Numbers



- Complex numbers in $\mathbb{C}$ can be mapped onto a plane which is called **the complex plane**.
- The coordinates of $z \in \mathbb{C}$ given by $z = z_{real} + jz_{imaginary}$ on the complex plane are $(z_{real}, z_{imaginary})$.
- The absolute value (or modulus) of z is the length from the origin: $|z| = \sqrt{(z_{real})^2 + (z_{imaginary})^2}$.
- The phase (or argument) of $z \in \mathbb{C}$ is found using a trigonometric function called the arctangent: $\angle z = \arctan(z_{imaginary}, z_{real})$.
- **Euler's formula** explicitly relates trigonometric functions and complex numbers:

$$e^{jx} = \cos(x) + j\sin(x) \qquad (1)$$

- Thus, $z = z_{real} + jz_{imaginary}$ and $z = |z|e^{j\angle z}$. The second representation is called the polar representation of z.
- The red vector representing a complex number is also called its **phasor** representation.

As we order the real numbers in $\mathbb{R}$ into a line, we can order the imaginary numbers into the imaginary line based on the real factor $b \in \mathbb{R}$ in $a = j\,b$. Let us consider how to collate complex numbers. Because for a $z \in \mathbb{C}$ the real and imaginary parts are arbitrary, and both have their own line, we can place each complex number on a plane that has real line as the horizontal axis and the imaginary line as the vertical axis. This is called the complex plane; the real and imaginary parts of $z = z_{real} + jz_{imaginary}$ give the Cartesian coordinates of z on this plane as $(z_{real}, z_{imaginary})$.

Complex numbers also have polar representations on the complex plane whereby the distance from the origin and angle with respect to positive real axis as the reference give the point's location. The distance from the origin is referred to as the absolute value (or modulus) of z and found using the Pythagoras' Theorem. The angle x of the complex number z is also called its phase or argument, and can be found using the inverse tangent function $arctangent(z_{real}, z_{imaginary})$ (see, https://en.wikipedia.org/wiki/Atan2).

Leonhard Euler's formula reveals the relations between trigonometric functions and the complex exponential function e^{jx}. Here, e is the base of the natural logarithm and also referred to as Euler's number. e raised to the complex power of jx equals to the complex number given by $\cos(x) + j\sin(x)$. This formula also relates the canonical representation of a complex number to its polar representation via $z = z_{real} + jz_{imaginary} = |z|e^{j\angle z}$.

The red vector is a graphical representation of z. Such vector representations of complex numbers are called phasors.

For further details on Euler's formula, see https://en.wikipedia.org/wiki/Euler%27s_formula.

For a slightly wider perspective on taking e to a complex power, see the short clip here: https://www.youtube.com/watch?v=v0YEaeIClKY

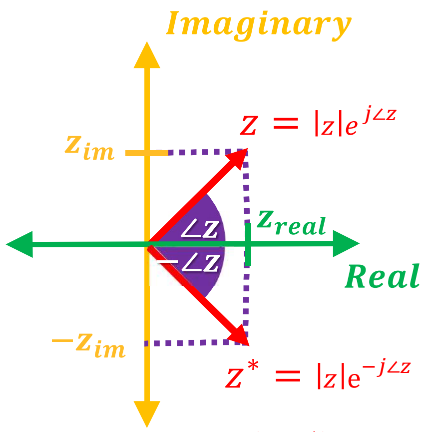
2.B.3 Complex Numbers

- The **complex conjugate** of a complex number $z = z_{real} + jz_{im} = |z|e^{j\angle z}$
  1. Has the same absolute value as z,
  2. Has a phase that is equal in magnitude to z's phase but opposite in sign,
  3. Denoted by z^*: $z^* = |z|e^{-j\angle z}$.

- Following **Euler's formula** in (1), and that the cosine and sine are even and odd functions, respectively, one gets

$$
\begin{aligned}
z^* &= |z|\cos(-\angle z) + j\,|z|\sin(-\angle z) \\
&= |z|\cos(\angle z) - j\,|z|\sin(\angle z) \\
&= z_{real} - jz_{im}
\end{aligned}
$$

i.e. $Re\{z^*\} = Re\{z\}, Im\{z^*\} = -Im\{z\}$.



Conjugate pair (z, z^*)

- In the summation of z with its complex conjugate z^*, imaginary parts cancel each other out and the result is twice the real part: $z + z^* = 2 \times Re\{z\}$
- The multiplication of z with its complex conjugate z^* equals to its absolute value squared: $z \times z^* = |z|^2$

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The complex conjugate of a complex number z is also a complex number and has the same absolute value as z and a phase equal in magnitude but opposite in sign. It is often denoted by z^* where superscript * denotes the conjugation operation, i.e. multiplying the phase of z in its polar form with -1 (see the top-right figure). Equivalently, if one considers the canonical (or conventional) form of z, its conjugate z^* is the complex number with an equal real part and an imaginary part equal in magnitude but opposite in sign.

In general, z and z^* form a conjugate pair; the complex conjugate of z^* is z , i.e. $(z^*)^* = z$.

When complex conjugate pairs are added up, their imaginary parts cancel each other out yielding a real valued number that equals to twice the real part of (any of the number in) the pair.

Multiplication of conjugate pairs similarly result with a real-valued number which equals to the squared absolute value (any of the number in) the pair.

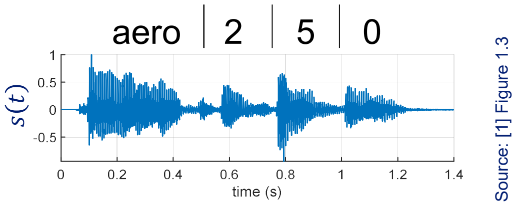
Note that, there are functions that map complex numbers to complex numbers. For example, a polynomial $g: \mathbb{C} \rightarrow \mathbb{C}$ when evaluated for complex numbers results with complex numbers and is a complex function. The fundamental theorem of algebra states that there are N complex roots of a polynomial of degree N. For example, the roots of the example polynomial in Section 1.B are (approximately) 0.7+j1.4 , 0.7-j1.4 and -0.39.

For further details on complex numbers, see your engineering mathematics and calculus textbooks and https://en.wikipedia.org/wiki/Complex_number .

An excellent series of video-lectures are here:
https://youtube.com/playlist?list=PLMrJAkhIeNNQBRslPb7I0yTnES981R8Cg

3 Signals: Definition

- Signals are functions of time and may describe a wide variety of phenomenon. E.g.
  - The air speed sensor readings over time.
  - A microphone's electrical signal at the output.
  - The electrical output of an antenna.
  - A video is an audio-visual signal.



- A real valued signal $s: \mathbb{R} \to \mathbb{R}$ maps a time value t to a real number. A complex signal $s: \mathbb{R} \to \mathbb{C}$ outputs a complex number s(t).
- The energy of a signal is given by

$$E = \int_{-\infty}^{\infty} |s(t)|^2 dt \qquad (2)$$

- If E is finite, i.e. $E < \infty$, $s(t)$ is referred to as an **energy signal**.
- The power of a signal is given by

$$P = \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} |s(t)|^2 dt \qquad (3)$$

- If P is non-zero and finite, i.e. $0 < \mathrm{P} < \infty$, $s(t)$ is a **power signal**.

Signals are functions of time that convey some useful information. For example, a sensor measurements over time, such as the output of an air speed sensor on an aircraft, is a signal. A microphone is a sensor that converts acoustic pressure to electrical "signals." The electrical output of an antenna is a signal. A video is an audio-visual signal…

Signals can be real valued or complex valued. We can classify signals as energy or power signals. Let us denote a signal by $s(t)$ and remind that power is the energy per unit time. The energy of a signal is given in (2). If this integral is finite, i.e. $E < \infty$, then the signal is referred to as an energy signal. If the energy integral in (2) is not finite, then, first we should restrict the bounds of the time interval we are calculating the energy for to some $-L$ and L. Second, we scale the energy term with the length of the integration interval, i.e. $2L$. Thus, now power is computed as the energy term is scaled with time. As different intervals might have different power values, we are interested in the overall average, hence take the limit in (3) as L tends to infinity. If the limit exists and is finite, then $s(t)$ is a power signal. For example, periodic signals are power signals which are discussed next.

Real-life signals of interest are always energy signals as no signal spans the entire time axis. However, it might still be an impractically large value and an average power is numerically more sensible especially when we deal with periodic signals. For example, the total electrical energy consumed by a kettle throughout its operational life-time is a finite number. Nevertheless, the average power of the (periodic) line signal that was drawn from the grid when working is a more useful quantity. There is more information on this in the remaining part of these notes.

4.A.1 Periodic Signals

- A signal is periodic with fundamental period T if it equals to the concatenation of a T long non-periodic segment of itself.
- Mathematically, a signal $s(t)$ is periodic with fundamental period T if for all t and integer m

$$s(t+mT)=s(t). \quad (4)$$

- E.g. The complex exponential signal

$$s(t)=e^{j2\pi\frac{1}{T}t} \quad (5)$$

is periodic with fundamental period T.

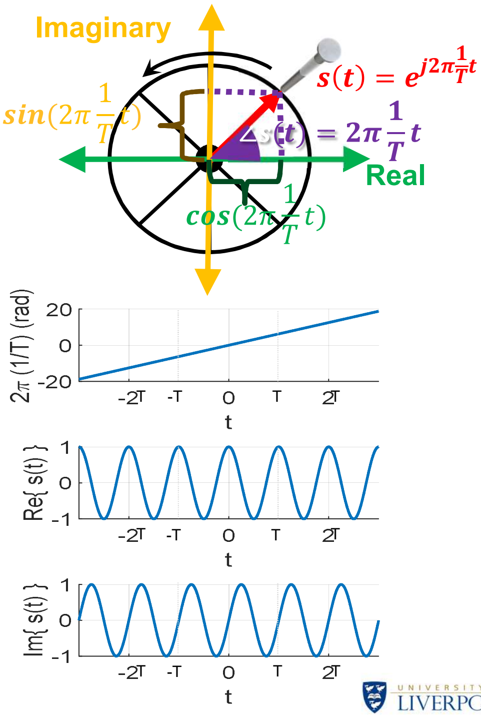
Following Euler's formula (1):

- The real part of $s(t)$ is

$$Re\{s(t)\}=\cos(2\pi\frac{1}{T}t)$$

- The imaginary part of $s(t)$ is

$$Im\{s(t)\}=\sin(2\pi\frac{1}{T}t)$$



Periodic signals are very common in avionics and communication systems. A signal is periodic if it repeats a pattern over time. Mathematically, a signal $s(t)$ is periodic with the fundamental period T if (4) is true for all $t \in \mathbb{R}$ and integer m, and T is the smallest of all such periods. The last condition is set out because if a signal is periodic with T, then it is periodic with $2T, 3T, \ldots$

One of the most commonly used periodic signals is the complex exponential signal given by (5). To see the periodicity, use (1) to expand (5) as

$e^{j2\pi\frac{1}{T}t}=\cos\left(2\pi\frac{1}{T}t\right)+j\sin(2\pi\frac{1}{T}t)$. The real and imaginary parts of this complex function are nothing but the coordinates of a nail's tip attached on a spinning wheel of radius one as shown in Section 2.A. In other words, we model the coordinates of a nail on a spinning wheel by using a complex function of time called the complex exponential. Because the nail's tip is going to be at the same position periodically with a period that equals to the spinning period of the wheel, the real and imaginary parts of the complex exponential are also periodic with T. As an exercise, show that

$s(t)=e^{j2\pi\frac{1}{T}t} \Rightarrow s(t+mT)=e^{j2\pi\frac{1}{T}t}$ for all $t \in \mathbb{R}$ and integer m (Hint: substitute $t+mT$ in $e^{j2\pi\frac{1}{T}t}$, use (1) and show that $\cos\left(2\pi\frac{1}{T}t\right)$ and $\sin\left(2\pi\frac{1}{T}t\right)$ are periodic with the fundamental period of T).

The second important property of the complex exponential is that its absolute value is one, i.e. $|s(t)|=1$. Thus, in the energy and power formulae in (2) and (3) $|s(t)|^2=1$. As a result, E diverges and P converges to 1, i.e. the complex exponential is a power signal. Complex exponential signals are also referred to as harmonic functions.

The graph of the function that maps $t \in \mathbb{R}$ to the angle value $2\pi\frac{1}{T}t$ is linear and passes through the origin; the real and the imaginary parts of the complex exponential signal are composite functions of this mapping and cosine and sine. Because the latter trigonometric functions are periodic in their argument with 2π, the complex exponential function is periodic with T.

There are other periodic functions which have graphs different than that of a sinusoidal function, e.g. see the graph on the top right. A practically important family of T periodic functions are related to complex exponential functions with fundamental frequencies $0, T, 2T, 3T, \ldots$ This point will be clarified later in this section.

4.A.2 Periodic Signals

- Let us multiply a complex exponential with a complex number α (or, weight factor) given by

$$\alpha = \alpha_{Re} + j\alpha_{Im}$$
$$= |\alpha| e^{j\angle\alpha}$$

- The weighted signal $w(t)$ is also periodic with fundamental period T:

$$w(t) = \alpha s(t), \quad = |\alpha| e^{j2\pi\frac{1}{T}t + \angle\alpha} \qquad (6)$$

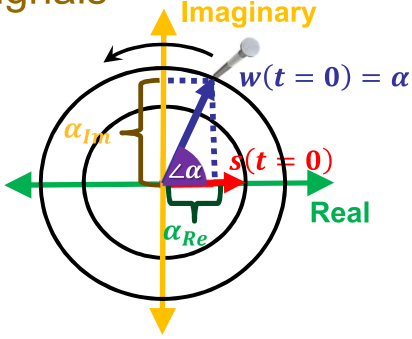
Following (1)

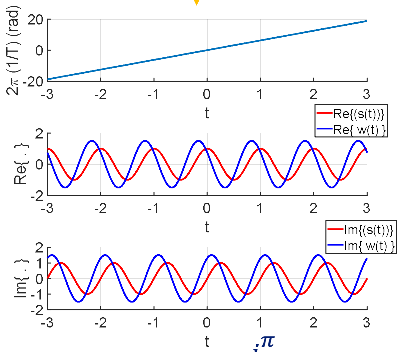
- The real part of $w(t)$ is

$$Re\{w(t)\} = |\alpha| \cos(2\pi \frac{1}{T} t + \angle\alpha)$$

- The imaginary part of $w(t)$ is

$$Im\{w(t)\} = |\alpha| \sin(2\pi \frac{1}{T} t + \angle\alpha)$$





E.g. $\alpha = 1.5e^{j\frac{\pi}{3}}$, $T = 1$

Let's first consider multiplying the complex exponential in (5) with a complex number, in other words, weighting it with a complex weight factor. Using the polar form of complex numbers shown in Section 2.B, any complex weight factor α (read alpha) is going to have the form $|\alpha|e^{j\angle\alpha}$ where $|\alpha|$ is the absolute value of α and $\angle\alpha$ is its phase. Now consider the complex exponential of period T at time $t = 0$, i.e. $s(0)$. The absolute value $|s(0)|$ is one and the phase $\angle s(0)$ is zero. Therefore, the weighted complex exponential given in (6) will have the same absolute value and the phase as α. In other words, at $t = 0$, the weighted exponential equals to α.

The resulting changes in the amplitude and phase of the graphs depicting the real and the imaginary parts of w are given on the slide for an example where $\alpha = 1.5e^{j\frac{\pi}{3}}$. The red curves are the real and imaginary parts of $s(t)$ which deviate between -1 and 1 as its absolute value is one. The blue curves are the real and imaginary parts of the weighted complex exponential $w(t)$ which deviate between -1.5 and 1.5 as the absolute value of $w(t)$ equals to that of α which, in this example, is 1.5. The cosine and the sine functions now have an additional phase term $\angle\alpha$ due to the weight factor: at time $t = 0$, the real part of $w(t)$ is not at its peak unlike $s(t)$ due to $\angle\alpha$. Similarly, its imaginary part does not evaluate to zero for this phase term.

As a result, the weight factor α shifts the complex exponential in time by its phase, and readjusts its magnitude by its absolute value.

4.A.3 Periodic Signals

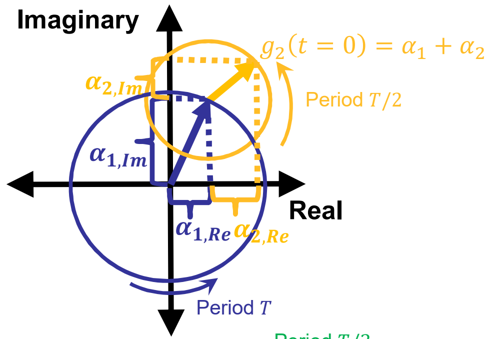
- Let us consider the superposition of two weighted complex exponentials the first with a fundamental period T (**fundamental frequency** $f = 1/T$), and the second with a fundamental period $\frac{T}{2}$ (frequency $2f$):

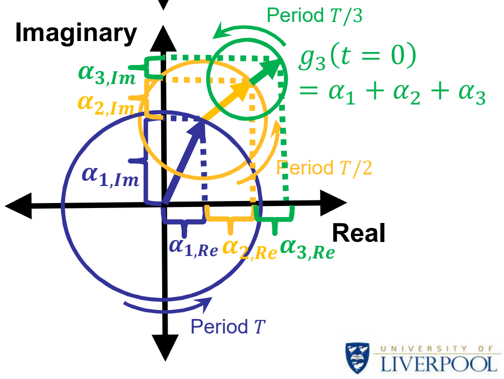
$$g_2(t) = \alpha_1 e^{j2\pi\frac{1}{T}t} + \alpha_2 e^{j2\pi\frac{1}{T/2}t}$$

Note: The vector representations on the right-hand side are also called **phasors.**

- Let us find the superposition of three weighted complex exponentials the first with a fundamental period T (frequency $f = 1/T$), the second with a fundamental period $T/2$ (frequency $2f$), and the third with a fundamental period $\frac{T}{3}$ (frequency $3f$):

$$g_3(t) = \alpha_1 e^{j2\pi\frac{1}{T}t} + \alpha_2 e^{j2\pi\frac{1}{T/2}t} + \alpha_3 e^{j2\pi\frac{1}{T/3}t}$$





As a second step, now consider taking the superposition of multiple weighted complex exponentials by summing them up. This way, a larger variety of periodic signals can be obtained. In the case of two components in superposition, the resulting signal will be in the form of $g_2(t)$ on the slide, i.e. $g_2(t)$ is a sum of two weighted complex exponentials with fundamental period T and $T/2$. Therefore, it is periodic with period T. This signal can be viewed as capturing the coordinates of a point on a circle which rotates on another circle. This is illustrated on the top right figure: the real-axis coordinate of the point (tip of the yellow vector) will also be the sum of the real parts of α_1 and α_2 at time $t = 0$. The imaginary-axis coordinate of this point is the sum of the imaginary parts of the α_1 and α_2 at the same time instant. In other time instants, these coordinates will be the sum of the real and imaginary parts of the weighted complex exponentials.

Let us now consider finding the superposition of three weighted complex exponentials with fundamental periods of T, $T/2$ and $T/3$, respectively. Equivalently, these are complex exponentials of frequency $\mathrm{f} = \frac{1}{T}$, 2f and 3f. The resulting signal is denoted by $g_3(t)$ where the subscript 3 denotes that a superposition of three components is used. This signal has a fundamental period of T, and illustrated on the bottom right figure. By using three weighted components, we can capture a larger variety of signals than can be captured by only two components.

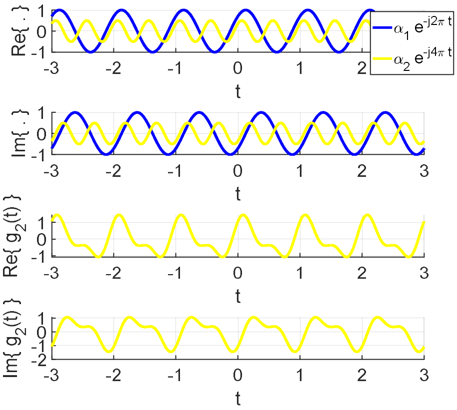
Remember from Section 2.A that the de facto alternative way of representing functions of time is to list ordered pairs which in this case would lead to the impractical number of items equalling to the size of the set $\mathbb{R}$. On the other hand, the weighted sums of complex exponentials provide a feasible alternative for the representation of a subset of all possible functions with a fundamental period of T based on known periodic functions with periods of $T, \frac{2}{T}, \ldots$

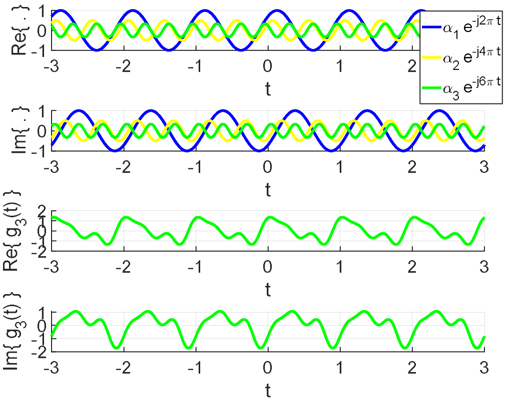
The circle-upon-circle structures on the slide are called epicycle and were first used by the ancient Greeks to study the orbits of planets (https://en.wikipedia.org/wiki/Deferent_and_epicycle).

4.A.4 Periodic Signals

- The graphs on the right-hand side are for
  - $T = 1$ s,
  - $\alpha_1 = 1e^{-j\frac{\pi}{4}}$
  - $\alpha_2 = \frac{1}{2}e^{-j\frac{\pi}{4}}$

  in finding $g_2(t)$.





- The graphs on the left-hand side are for
  - $T = 1$ s,
  - $\alpha_1 = 1e^{-j\frac{\pi}{4}}$
  - $\alpha_2 = \frac{1}{2}e^{-j\frac{\pi}{4}}$
  - $\alpha_3 = \frac{1}{3}e^{j\frac{\pi}{4}}$

  in finding $g_3(t)$.

In a two-component superposition example, assume that the fundamental period $T = 1$ s and the (complex) weights are as follows: the first weight's absolute value is 1, and its phase is $-\pi/4$. The second weight has the same phase as the first weight (i.e. the phasors in the top right figure of the previous slide are aligned at $t = 0$), but its magnitude is half as the first one. The real and imaginary parts of these components of $T = 1$ and $T = 1/2$ are depicted in the top-right figure. The resulting two component superposition $g_2(t)$ is depicted in the third and the fourth tabs as the graphs of its real and imaginary parts over time, respectively.

Continuing with the same example for the three component superposition $g_3(t)$, we select a third weight α_3 that has the negative phase of the other weights, i.e. $\pi/4$, and a magnitude of 1/3. The real and imaginary parts of the resulting third weighted component of period $T = 1/3$ is depicted in the bottom-left figure together with the first two components. The resulting superposition's real and imaginary parts are very different compared to the graphs of the two component case.

When $\alpha_3 = 0$ the three component superposition equals to the two component superposition, and by varying α_3 signals that could not be represented by only two components can now be represented. Therefore, addition of more weighted complex exponentials extend the set of periodic functions that can be represented by using a weighted sum of complex exponentials.

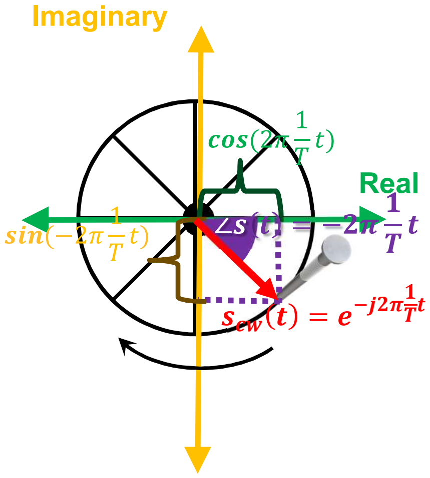
4.A.5 Periodic Signals

- Let us consider a **clockwise** complex exponential with a fundamental period T

$$s_{cw}(t) = e^{-j2\pi\frac{1}{T}t} \qquad (7)$$

- $s_{cw}(t)$ is the **complex conjugate** of the complex exponential $s(t)$ in (5):

$$Re\{s_{cw}(t)\} = Re\{s(t)\} = \cos\left(2\pi\frac{1}{T}t\right),$$
$$Im\{s_{cw}(t)\} = -Im\{s(t)\} = -\sin\left(2\pi\frac{1}{T}t\right).$$



- The frequency of $s_{cw}(t)$ is $f = -1/T$ (**negative frequency** indicating reverse rotation with respect to the reference).

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Before going into further details of the set of periodic functions that can be written as a superposition of weighted complex exponentials, let us introduce a second type of complex exponential which represents clockwise rotations. This function is given in (7) and is the complex conjugate of the complex exponential in (5) for all $t \in \mathbb{R}$.

Because the sign of the factor getting multiplied with the time variable t is negative, $s_{cw}(t)$ is sometimes referred to as a complex exponential with "negative frequency"; this term sometimes might be misleading. The "negative" quantity here is related to that a clockwise rotation of a point corresponds to the angle of the phasor with respect to the positive real axis taking negative values and decreasing by the rotation as t grows.

Euler's formula when used with (7) yields

$$s_{cw}(t) = \cos\left(-2\pi\frac{1}{T}t\right) + j\sin\left(-2\pi\frac{1}{T}t\right).$$

In Section 2.A, we showed that cosine is an even function and sine is an odd function. In other words, $\cos\left(-2\pi\frac{1}{T}t\right) = \cos\left(2\pi\frac{1}{T}t\right)$ and $\sin\left(-2\pi\frac{1}{T}t\right) = -\sin\left(2\pi\frac{1}{T}t\right)$. As a result

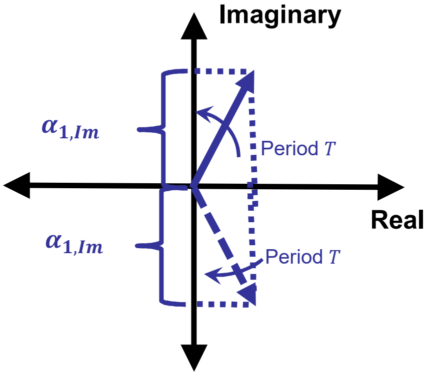
$$s_{cw}(t) = \cos\left(2\pi\frac{1}{T}t\right) - j\sin\left(2\pi\frac{1}{T}t\right).$$

4.A.6 Periodic Signals

- Now, let's consider the superposition of a pair of weighted complex exponentials such that they constitute a complex conjugate pairs, i.e.
  - $w_1(t) = \alpha_1\, e^{j2\pi\frac{1}{T}t}$,
  - $\left(w_1(t)\right)^* = \alpha_1^*\, e^{-j2\pi\frac{1}{T}t}$

  and,

$$g(t) = w_1(t) + \left(w_1(t)\right)^*$$



- Following 2.B.3
  - $w_1(t) + \left(w_1(t)\right)^* = 2Re\{w_1(t)\}$.
- Following 4.A.2
  - $Re\{w_1(t)\} = |\alpha_1| \cos\left(2\pi\frac{1}{T}t + \angle\alpha_1\right)$.
- The figure shows that $x(t)$ is real valued and can be found as (see slide 2.B.3)

$$g(t) = 2|\alpha_1| \cos\left(2\pi\frac{1}{T}t + \angle\alpha_1\right)$$

- Two complex exponentials with complex conjugate weights sum up to yield a sinusoidal function!

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Now, consider the superposition of a pair of weighted complex exponentials such that they constitute a complex conjugate pair. In other words, if one of the components is selected as $w_1(t) = \alpha_1\, e^{j2\pi\frac{1}{T}t}$ (with fundamental period T), then the second component is its complex conjugate $\left(w_1(t)\right)^* = \alpha_1^*\, e^{-j2\pi\frac{1}{T}t}$. Now let us consider $w_1(t) + \left(w_1(t)\right)^*$. From slide 2.B.3, we know that this summation is twice the real part of $w_1(t)$(or equivalently, $\left(w_1(t)\right)^*$). Therefore, we need to find $2Re\{w_1(t)\}$. The real part of $w_1(t)$ is the first term on the right-hand side of the last equality below obtained after some rearrangements:

$$w_1(t) = |\alpha_1| e^{j\angle\alpha_1} e^{j2\pi\frac{1}{T}t}$$
$$= |\alpha_1| e^{j(2\pi\frac{1}{T}t + \angle\alpha_1)}$$
$$= |\alpha_1| \cos\left(2\pi\frac{1}{T}t + \angle\alpha_1\right) + j|\alpha_1| \sin\left(2\pi\frac{1}{T}t + \angle\alpha_1\right)$$

As a result, $2Re\{w_1(t)\} = 2|\alpha_1| \cos\left(2\pi\frac{1}{T}t + \angle\alpha_1\right)$.

Therefore, two complex exponentials with complex conjugate weights sum up to yield a sinusoidal function!

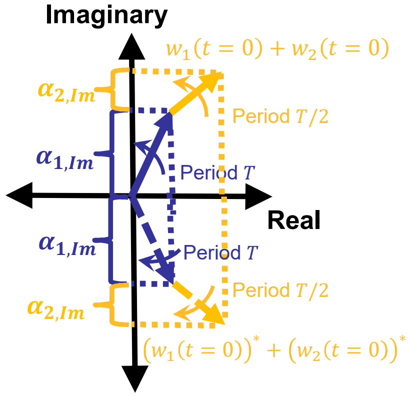
4.A.7 Periodic Signals

- Now, let's consider the superposition of four weighted complex exponentials such that they constitute two complex conjugate pairs, i.e. define
  - $w_1(t) = \alpha_1\, e^{j2\pi\frac{1}{T}t}$,
  - $w_2(t) = \alpha_2\, e^{j2\pi\frac{1}{T/2}t}$.

  and find

$$g(t) = w_1(t) + \left(w_1(t)\right)^* + w_2(t) + \left(w_2(t)\right)^*.$$

- Following 2.B.3
  - $\left(w_1(t)\right)^* = \alpha_1^*\, e^{-j2\pi\frac{1}{T}t}$,
  - $\left(w_2(t)\right)^* = \alpha_2^*\, e^{-j2\pi\frac{1}{T/2}t}$,
  - $w_1(t) + \left(w_1(t)\right)^* = 2Re\{w_1(t)\}$.
- Following 4.A.2
  - $Re\{w_1(t)\} = |\alpha_1|\cos\left(2\pi\frac{1}{T}t + \angle\alpha_1\right)$.
- The figure shows that $x(t)$ is real valued and can be found as (see slide 2.B.3)



$$g(t) = 2|\alpha_1|\cos\left(2\pi\frac{1}{T}t + \angle\alpha_1\right) + 2|\alpha_2|\cos\left(2\pi\frac{2}{T}t + \angle\alpha_2\right).$$

Next, consider the superposition of four weighted complex exponentials such that these components constitute two complex conjugate pairs. In other words, if two components are selected as $w_1(t) = \alpha_1\, e^{j2\pi\frac{1}{T}t}$ and $w_2(t) = \alpha_2 e^{j2\pi\frac{1}{T/2}t}$(with fundamental period T and $T/2$), then the third and fourth component are their complex conjugates $\left(w_1(t)\right)^*$=$\alpha_1^*\, e^{-j2\pi\frac{1}{T}t}$ and $\left(w_2(t)\right)^* = \alpha_2^*\, e^{-j2\pi\frac{1}{T/2}t}$. Now let us consider $w_1(t)$+ $\left(w_1(t)\right)^*$. From slide 2.B.3, we know that this summation is twice the real part of $w_1(t)$(or equivalently, $\left(w_1(t)\right)^*$). Therefore, we need to find $2Re\{w_1(t)\}$. The real part of $w_1(t)$ is the first term on the right-hand side of the last equality below obtained after some rearrangements:

$$w_1(t) = |\alpha_1| e^{j\angle\alpha_1} e^{j2\pi\frac{1}{T}t}$$
$$= |\alpha_1| e^{j(2\pi\frac{1}{T}t + \angle\alpha_1)}$$
$$= |\alpha_1|\cos\left(2\pi\frac{1}{T}t + \angle\alpha_1\right) + j|\alpha_1|\sin\left(2\pi\frac{1}{T}t + \angle\alpha_1\right)$$

As a result, $2Re\{w_1(t)\} = 2|\alpha_1|\cos\left(2\pi\frac{1}{T}t + \angle\alpha_1\right)$.

The summation of the remaining terms are similarly $w_2(t)$+ $\left(w_2(t)\right)^* = 2Re\{w_2(t)\}$ and one can similarly show that $2Re\{w_2(t)\} = 2|\alpha_2|\cos\left(2\pi\frac{1}{T}t + \angle\alpha_2\right)$. The superposition of these four weighted complex exponentials is a real-valued signal that is periodic with T.

Note that by superposing more complex exponentials and varying their weights we can represent a wider variety of periodic signals.

4.B.1 Fourier Series

- Let us generalise the weighted sum approach to an (infinite) number of complex exponentials in a weighted superposition to **synthesise** a signal:

$$g(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{j2\pi\frac{n}{T}t} \quad (8)$$

- **The number of the complex exponentials:** There are as many complex exponentials in (8) as there are numbers in the set of integers $\mathbb{I}$.
- **The periodicity of the complex exponentials**: All complex exponentials are periodic with T (with fundamental period T/n for $\mathrm{n} \neq 0$).
  For $\mathrm{n} = 0$, the complex exponential is a constant function.
- **The frequency of the complex exponentials**: The frequency of the nth complex exponential is $f_n = \frac{n}{T}$.
  Half of the complex exponentials have positive frequencies ($f = n/T$ for $n = 1,2, \ldots$) and half of the complex exponentials have negative frequencies ($f = n/T$ for $n = -1, -2, \ldots$). For $\mathrm{n} = 0$ the frequency $f = 0$ (indicating a constant function).
- **The weights of the complex exponentials**: The complex coefficients α_n can be any (finite) complex number that yield a converging sum.
- The equality in **(8)** is **The Fourier Series Synthesis Formula** named after Joseph Fourier and can represent all signals of practical interest with a fundamental period T. The weights α_n are called **The Fourier Series coefficients.**

Let us generalise the weighted sum of complex exponentials approach by using all integer multiples of the fundamental frequency $f = \frac{1}{T}$. This leads to the (infinite) series sum in (8) where α_ns are complex weights and $e^{j2\pi\frac{n}{T}t}$

is a complex exponential with a fundamental period of T/n.

In this summation, there are as many complex exponentials as there are numbers in the set of integers $\mathbb{I}$. All of the complex exponentials in the summation are periodic with T and each has a fundamental period of T/n for $\mathrm{n} \neq 0$. Thus, half of the complex exponentials have positive frequencies ($f_n = n/T$ for $n = 1,2, \ldots$) and half of the complex exponentials have negative frequencies ($f = n/T$ for $n = -1, -2, \ldots$). The complex coefficients α_n can be any (finite) complex number that yield a converging sum.

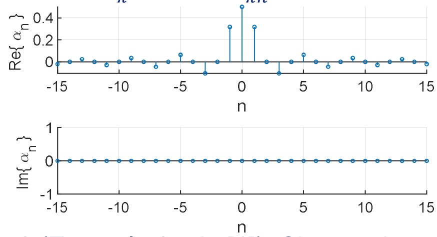
A very remarkable property of (8) is that all signals of practical interest with a fundamental period T can be written as the right-hand side of (8); the equality holds when the coefficients are selected accordingly. In other words, by varying α_ns one can synthesise any signal of fundamental period T.

In fact, (8) is The Fourier Series Synthesis Equation named after Joseph Fourier who developed the idea of representing periodic functions with series of trigonometric functions in the early 19th century. In this context, the weights α_n are referred to as The Fourier Series coefficients.

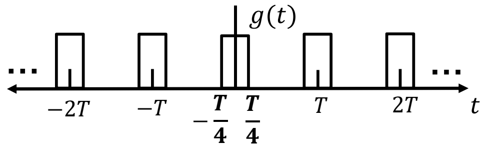
4.B.2 Fourier Series

- Example 1 (Example 3.5 in [1]): Choose the weights as follows:

$$\alpha_0 = \frac{1}{2},\ \alpha_1 = \alpha_{-1} = \frac{1}{\pi}, \ldots, \alpha_n = \frac{\sin\left(\frac{\pi n}{2}\right)}{\pi n}, \ldots$$

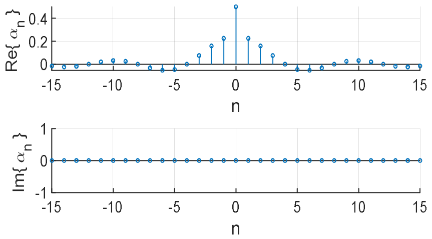


Eq. (8) synthesise a pulse train with a fundamental period of T and pulse width of **T/2**.

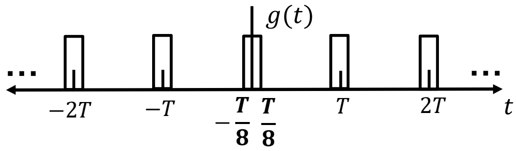


- Example 2 (Example 3.5 in [1]): Choose the weights as follows:

$$\alpha_0 = \frac{1}{2},\ \alpha_1 = \alpha_{-1} = \frac{\sqrt{2}}{2\pi}, \ldots, \alpha_n = \frac{\sin\left(\frac{\pi n}{4}\right)}{\pi n}, \ldots$$



Eq. (8) synthesise a pulse train with a fundamental period of T and pulse width of **T/4**.



On this slide are two examples on how selection of different weights yield different signals:

In the first example, the weights (or Fourier coefficients) are selected as $\alpha_n = \frac{\sin\left(\frac{\pi n}{2}\right)}{\pi n}$. These weights are real-valued, i.e. their imaginary part is zero. The Fourier Series Synthesis Equation in (8) results with a pulse train of fundamental period T depicted in the top-right figure. This signal is real-valued and symmetric with respect to the vertical axis (i.e., it is an even signal). The pulse width is $\frac{T}{4} + \frac{T}{4} = \frac{T}{2}$.

When the weights are selected slightly differently as $\alpha_n = \frac{\sin\left(\frac{\pi n}{4}\right)}{\pi n}$, all properties of the synthesised signal using (8) remain the same except the pulse width. By changing the coefficients, we now have a pulse width of $\frac{T}{8} + \frac{T}{8} = \frac{T}{4}$.

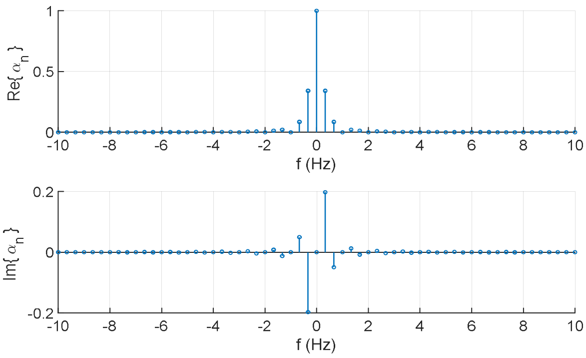
In Fourier series, the n^{th} coefficient is associated with trigonometric functions of frequency $f_n = \frac{n}{T}$. Therefore, the left-hand side figures reveal the “frequency content” of the signals on the right-hand side.

One remarkable point here is that discontinuous/sharply changing signals such as rectangular pulse trains are synthesised using weighted superpositions of smooth trigonometric functions (see also Section 4.A and Equation (6)).

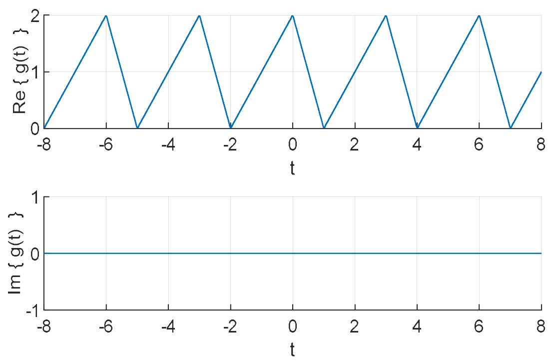
Radars and certain communication systems use pulse trains that are similar to the ones in these examples.

4.B.3 Fourier Series

- Example 3 (Problem 3.22 in [1]): Choose the weights as follows (with non-zero imaginary parts):



Eq. (8) synthesise an asymmetric sawtooth of fundamental period T=3:



- Note that each α_n is associated with frequency $f_n = n/T$. The top-left figure has these frequencies along the horizontal axis replacing n to highlight the interchangeability.
- Also note that the horizontal axis is discrete, i.e., frequencies other than f_n are not elements of this axis – they are not accounted for.

Now, let us look at an example in which the imaginary parts of the Fourier Series coefficients are not all zero and as given in the top-left figure. The synthesised signal using Equation (8) is on the right-hand side: an asymmetric sawtooth which is real-valued, i.e. the imaginary part of the synthesised signal is zero for all $t = 0$. The latter follows from that positive and negative indexed coefficients are complex conjugate pairs (see Section 4.A.6), i.e. $\alpha_n = \alpha^*_{-n}$. The fundamental period of this signal is $T = 3$. Therefore, the n^{th} coefficient weights trigonometric functions of frequency $f_n = n/3$. This relation is explicitly shown on the left hand side figure by depicting the real and imaginary parts against the frequencies they are associated with.

Although $f_n = \frac{n}{3}$ for $n \in \mathbb{I}$ are marked on the horizontal axis, other frequency values are not part of the Fourier series, i.e. as far as a signal with period T is concerned, terms with frequencies other than f_n are not taken into account. This point is going to become more clear during the discussion on aperiodic signals and the Fourier Transform later in these notes.

For an animated Fourier Series demonstration, see

https://www.youtube.com/watch?v=YUBe-ro89I4

For an audio demonstration, see

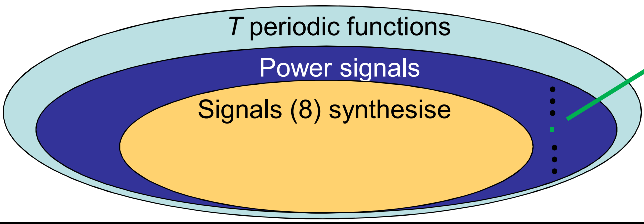
https://www.youtube.com/watch?v=3IAMpH4xF9Q

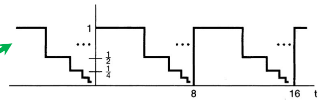
4.B.4 Fourier Series

- **The Fourier Series Analysis Formula** gives the weights to use in (8) and synthesise a desired signal $g(t)$ with a fundamental period of T

$$\alpha_n = \frac{1}{T}\int_{<T>} g(t)e^{-j2\pi\frac{n}{T}t}dt \qquad (9)$$

- Here, the integration is carried out over any time interval of length T.
- In other words, $\int_{<T>} dt = \int_{\tau}^{\tau+T} dt$ for some $\tau \in \mathbb{R}$. For example, τ can be selected as $\tau = 0$ or $\tau = -T/2$.
- The formula needs to be used for all $n = \cdots, -2, -1, 0, 1, 2, \ldots$
- All α_n must be finite to have a convergent sum in (8) (necessary but not sufficient condition). All "power signal" $g(t)$ that have "nice" behaviour yield convergent α_ns in (8).





Source: Figure 3.8 [1].

How can we find the coefficients to synthesise a signal $g(t)$ we are interested using The Fourier Series Synthesis Equation in Equation (8)? The answer is given by The Four Series Analysis Equation given in Equation (9): the nth Fourier coefficient equals the integration of the product of the nth complex exponential with $g(t)$ over a period of T scaled by T.

Integration over a period means the lower and upper limits of the integration can be selected as any time instants separated exactly by T. For example, one can choose the lower limit as 0 and upper limit as T and evaluate $\alpha_n = \frac{1}{T}\int_0^T g(t)e^{-j2\pi\frac{n}{T}t}dt$ to find the Fourier coefficients of $g(t)$. An equivalent alternative is to select the lower limit as –T/2 and the upper limit as T/2, i.e. evaluate $\alpha_n = \frac{1}{T}\int_{-T/2}^{T/2} g(t)e^{-j2\pi\frac{n}{T}t}dt$. The main reason all such choices are equivalent is because both $g(t)$ and the nth complex exponential are periodic with T.

Equation (9) finds how much $g(t)$ contains the nth complex exponential. Thus, we find the frequency content of $g(t)$ using the analysis formula in Equation (9) for all integers $n = \cdots, -2, -1, 0, 1, 2, \ldots$ Almost all periodic functions/power functions of practical interest can be analysed using equations (9)

There are certain T periodic functions, however, that cannot be analysed/synthesised using Equations (9) and (8). One example is depicted on the bottom right: This signal has an infinite number of discontinuities within each period of length T. Therefore, it is not a "nice" function as the pulse trains which had two discontinuities within a period. There are other examples in which one can define periodic functions that yields a divergent integral in (9), or a non-convergent sum in (8) for having infinitely many maximums/minimums or discontinuities within a period. Such signals do not admit Fourier Series representations and are out of the scope of our discussion.

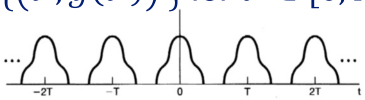
4.B.5 Fourier Series

- As a summary, **Fourier Series** represent periodical signals through **synthesis and analysis formulae** given in (8) and (9):

$$g(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{j2\pi\frac{n}{T}t} \leftrightarrow \alpha_n = \frac{1}{T}\int_{<T>} g(t)e^{-j2\pi\frac{n}{T}t}dt$$

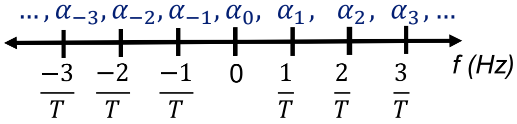
- A periodic power signal with period T (that also has “nice behaviour ”) has two equivalent representations: Its time domain representation (e.g. ordered pairs, its graph etc), and Fourier series coefficients.

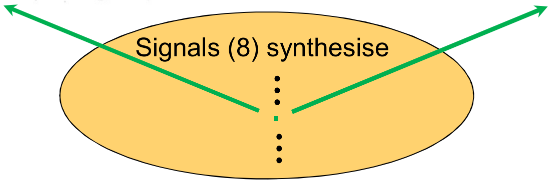
$\{(t', g(t'))\}$ for $t' \in [0, T)$



Source: [1] Figure 1.14

$\ldots, \alpha_{-3}, \alpha_{-2}, \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \alpha_3, \ldots$





As a summary, The Fourier Series representation of a well-behaving (“nice”) periodic signal $g(t)$ refers to: 1) A weighted sum of complex exponentials that is referred to as The Fourier Series Synthesis Formula and finds $g(t)$, and 2) The Fourier Series Analysis Formula to find the “Fourier coefficients” of $g(t)$ which are the weights of the complex exponentials in the synthesis formula.

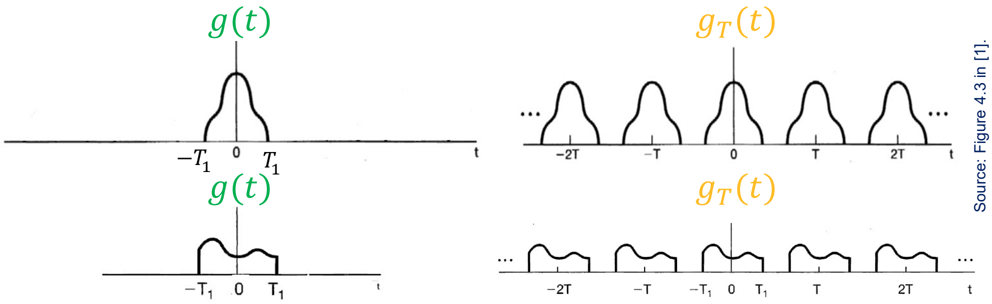
As a result, there are two equivalent representations of a signal with a fundamental period of T: the first is the so-called “time-domain representation” that consists of the graph of the function against time, or equivalently, an ordered list of time and function value pairs over a time interval of length T. The second representation is given by the Fourier series coefficients, which reveal the contents of $g(t)$ in terms of complex exponentials. This representation is a “frequency-domain representation” as each complex exponential is associated with a frequency. All signals that can be analysed and synthesised using (9) and (8) admit these two characterisations: Given one, the other can be found using the relevant Fourier Series equation.

Pointers to further reading on the topic can be found in the last page.

There are some excellent online content explaining and visualising Fourier Series. One particularly nice one is here: https://www.youtube.com/watch?v=r6sGWTCMz2k

5.A.1 Non-periodic Signals

- Non-periodic signals are non-repetitive functions of time.
- We are mainly interested in non-periodic energy signals (see Sec. 3).
- Below are examples of non-periodic functions and associated T periodic functions obtained by replicating them around integer multiples of T :



Non-periodic signals — T periodic counterparts

- Non-periodic energy signals on the left hand side can be viewed as the limit of their T-periodic counterpart as T tends to infinity.

$$\lim_{T\to\infty} g_T(t) \to g(t)$$

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Non-periodic signals are non-repetitive functions of time. In other words, there is no finite T such that (4) can be true. Energy signals (see Section 3) are typical non-periodic signals as most of the time they are zero for all time values except a certain time interval.

On the slide are examples of non-periodic signals: they take values of interest over $[-T_1, T_1]$ and are zero elsewhere. One can construct periodic counterparts from them by shifting them around integer multiples of $T > 2T_1$, i.e. $\dots, -3T, -2T, -T, 0, T, 2T, 3T$.

We can view non-periodic signals as the limit of their T periodic counterparts $g_T(t)$ as T is increased towards infinity. This view will be very useful when extending Fourier Series analysis and synthesis formulae to non-periodic signals.

5.B.1 The Fourier Transform

- The Fourier Transform reveals the weighted complex exponentials constituting a non-periodic signal in a similar way that The Fourier Series reveal it for periodic signals.
- Let us first consider The Fourier Series analysis and synthesis equations together:

$$g_T(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{j2\pi \frac{n}{T} t} \quad \text{where} \quad \alpha_n = \frac{1}{T} \int_{<T>} g_T(t) e^{-j2\ \frac{n}{T} t} dt.$$

1. Substitute the latter in the former: $g_T(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T} \left(\int_{<T>} g_T(t') e^{-j2\pi \frac{n}{T} t'} dt' \right) e^{j2\pi \frac{n}{T} t}$.

2. Define a function for a continuous frequency f as $G(f) = \int_{-\infty}^{\infty} g(t') e^{-j2\pi f t'} dt'$ (10) which equals to the term inside the parenthesis when evaluated at $f_n = \frac{n}{T}$. So,

$$g_T(t) = \sum_{n=-\infty}^{\infty} G(f_n) e^{j2\pi f_n t} \Delta f_n, \quad \text{where } \Delta f_n = f_{n+1} - f_n = \frac{1}{T}.$$

- The right hand side is a Reimann sum and as T tends to infinity $\Delta f_n \rightarrow df$, $\sum_{n=-\infty}^{\infty}(.)\Delta f_n \rightarrow \int_{-\infty}^{\infty}(.)df$ and we get:

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi f t} df. \ (11)$$

Is it possible to decompose non-periodic signals as a superposition of weighted complex exponentials in a way similar to using Fourier series for periodic signals to reveal their “contents”? You might guess that the answer is yes. To see this, let us begin with the periodic counterpart $g_T(t)$ of a non-periodic signal $g(t)$ and consider The Fourier Series synthesis and analysis equations (8) and (9), respectively, applied for the counterpart.

Now:

Substitute the left-hand side of the analysis equation in place of the Fourier coefficient α_n in the synthesis equation.

The equation from the first step involves an integration over one period T which can be selected as the interval $\left[-\frac{T}{2}, \frac{T}{2}\right]$, for example. Now, let's replace this integral with a slightly different one which evaluates exactly the same at the harmonic frequencies of the series, i.e. $f_n = \frac{n}{T}$. As a replacement, let's use the non-periodic counterpart of $g_T(t)$, i.e. $g(t)$, as the integrand and select the integration interval as the entire real line, i.e. $(-\infty, \infty)$. Also introduce a free variable f which can take any value and not only integer multiples of $1/T$. This newly defined function is in (10) and results with the same output as the integral inside the parentheses when evaluated at $f = f_n$ for all n; it can also map frequency values that are not integer multiples of $1/T$ (i.e. maps any $f \neq f_n$ to a complex number).

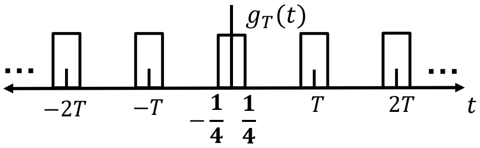
As a result, the rearranged equation features the periodic signal $g_T(t)$ on the left-hand side and a Reimann sum on the right hand side*. The step-size in the Reimann sum is $1/T$ and the function that is evaluated with $1/T$ steps is $G(f)e^{j2\pi f t}$. Therefore, as T tends to infinity, the left hand side approaches to the non-periodic signal $g(t)$, and the right-hand side approaches to the integral in (11).

The equations (10) and (11) are the Fourier Transform analysis and synthesis equations, respectively. Both $g(t)$ and $G(f)$ are complex functions of their real-valued arguments, in general (i.e. $g: \mathbb{R} \rightarrow \mathbb{C}$ and $G: \mathbb{R} \rightarrow \mathbb{C}$).

* See your engineering mathematics text book and lecture notes for Reimann sums or https://en.wikipedia.org/wiki/Riemann_sum

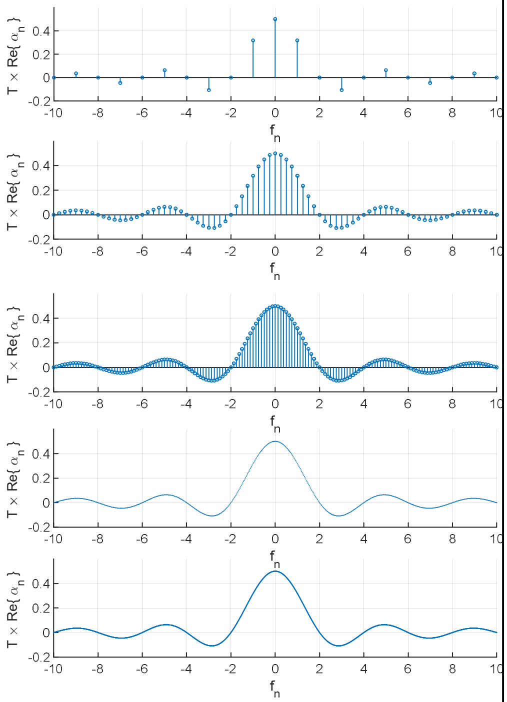
5.B.2 The Fourier Transform

- Example 4: Consider the pulse train example in Section 4.B.2.



- On the right hand side are The Fourier Series Coefficients α_n vs $f_n = \frac{n}{T}$ demonstrating how as T grows, f_ns becomes denser and $T \times \alpha_n$ tend to form a continuous function of the frequency f given in (10):

  from top to bottom
  $T = 1,4,8,32$ and $T = 256$.



Let us demonstrate how increasing T leads to more densely located terms on the frequency axis, and how the Fourier Series coefficients weighted with T are samples taken from the same function. This is because as T increases, the $1/T$ steps in-between the frequency terms become smaller and smaller.

As an example, we consider the pulse train in 4.B.2. That example demonstrated how changing the pulse-width vary the Fourier series coefficients. Here, we fix the pulse width to ½ and change the period T.

Specifically, we increase the period as $T = 1,4,8,32$ and $T = 256$. The resulting coefficients (weighted by T to remove the scaling in equation (9)) vs the frequency terms are given on the right hand side starting from the top. For $T = 1$, the frequency terms are $0, \pm 1, \pm 2, \ldots$. For $T = 4$, the frequency terms are now at $0, \pm\frac{1}{4}, \pm\frac{2}{4}, \pm\frac{3}{4}, 1, \pm\frac{5}{4}, \ldots$ For $T = 32$, the frequency terms are already dense enough to have a plot that appears as continuous. For $T = 256$ the plot is more smooth and appears more similar to a continuous function's graph: The resolution is now so high that the "dots" appear as a continuous curve.

The function whose graph is the envelope of all of the plots here is nothing but $G(f)$, i.e. the Fourier Transform of the non-periodic counterpart of $g_T(t)$. In the limit of T tending to infinity, we will have coverage of the entire "spectrum" of frequencies and the graph of the function $G(f)$ and the periodic function $g_T(t)$ will have become a non-periodic one. This limit case is the Fourier Transform.

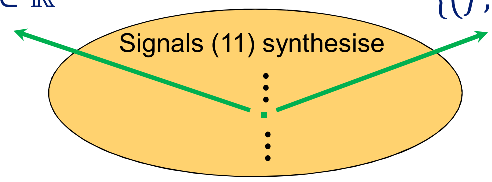
5.B.3 The Fourier Transform

- The Fourier Transform Synthesis and Analysis equations are as follows:

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft}df \quad \leftrightarrow \quad G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi} \quad dt$$

- A non-periodic energy signal (that also has “nice behaviour”) has two equivalent representations: Its time domain representation (e.g. ordered pairs, its graph etc), and its Fourier Transform.

$\{(t, g(t))\}$ for $t \in \mathbb{R}$ $\qquad$ $\{(f, G(f))\}$ for $f \in \mathbb{R}$



- $G(f)$ is called the **Fourier Transform** of $g(t)$ or its **spectrum**; $g(t)$ and $G(f)$ constitute a Fourier Transform pair.
- Some periodic signals such as the complex exponential and trigonometric signals also admit Fourier Transforms.

Thus, for a well-behaving (“nice”) non-periodic signal: 1) The Fourier Transform Synthesis Formula (left) expresses $g(t)$ as an infinite sum of weighted complex exponentials over a continuum of frequencies, and 2) The Fourier Transform Synthesis Formula (top-right) finds the “Fourier Transform” (or spectrum) of $g(t)$, which is a complex function of a continuous frequency variable denoted by $G(f)$, and weights the complex exponentials in the synthesis formula together with the infinitesimal df. In other words, the synthesis equation can be viewed as $g(t) = \int_{-\infty}^{\infty}(G(f)df)\, e^{j2\pi ft}$ where the term inside the parentheses is the weight of the complex exponential $e^{j2\pi ft}$ in an infinite sum.

As a result, there are two equivalent representations of a non-periodic signal revealed by the Fourier Transform: the first is the so-called “time-domain representation” that consists of the graph of the function against time, or equivalently, an ordered list of time and function value pairs over the real valued time axis. The second representation is given by the Fourier Transform, which reveal the contents of $g(t)$ in terms of complex exponentials in a way similar to the Fourier Series reveal the same for periodic signals. Also called the “spectrum” or “the frequency-domain representation” of $g(t)$, $G(f)$ maps a real valued frequency f to a complex number. All signals that can be analysed and synthesised using (10) and (11) admit these two characterisations: Given one, the other can be found using the relevant Fourier Transform equation.

Pointers to further reading on the topic can be found on the last page.

All energy functions have Fourier Transforms. Some periodic signals such as the complex exponential and trigonometric signals also admit Fourier Transforms which is going to be discussed later.

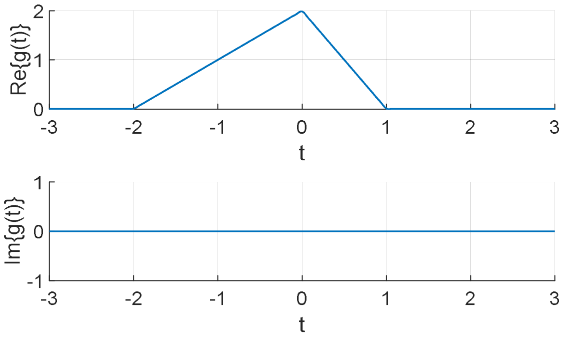
There are some excellent online content explaining and visualising the Fourier Transform. One particularly nice one is here: https://www.youtube.com/watch?v=spUNpyF58BY

5.B.4 The Fourier Transform

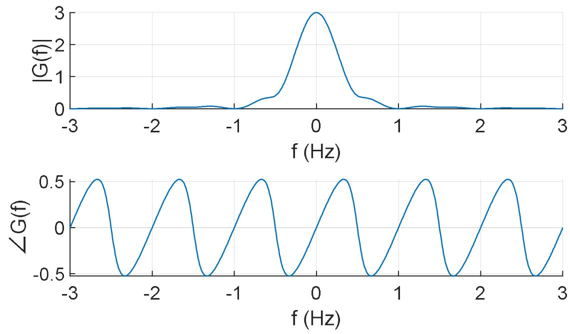
- Given **spectrum** $G(f)$, its absolute value is called the **amplitude spectrum** and its phase is called the **phase spectrum**.

$$G(f) = \underbrace{|G(f)|}_{\text{Amplitude Spectrum}} e^{\underbrace{\angle G(f)}_{\text{Phase Spectrum}}}$$

- Example 5: Consider the non-periodic counterpart of the sawtooth function in Example 3:



Eq. (10) finds the spectrum $G(f)$ depicted as amplitude and phase spectrums:

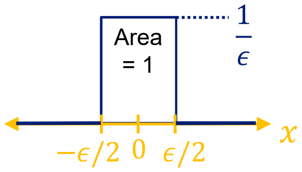


The Fourier Transform of a signal $G(f)$ is also referred to as its spectrum. As $G(f)$ is a complex number corresponding to f, so one can plot the graph of its real and imaginary parts versus frequency. Also, $G(f)$ has an absolute value (or magnitude, or amplitude) and a phase in its polar form (see Section 2.B.2). Therefore, one can plot its magnitude and the phase angle against frequency. The first plot is called the amplitude spectrum and the second plot is called the phase spectrum of $g(t)$.

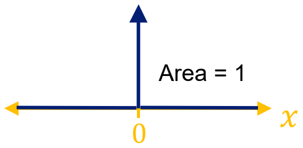
As an example, let us consider the non-periodic counterpart of the sawtooth signal in Section 4.B.3. This is a real valued signal in the form of an asymmetric triangle as depicted on the left-hand side. This is clearly an energy signal. The magnitude and phase spectrum of this signal obtained by first finding its Fourier Transform $G(f)$ and then finding the absolute value and the angle of $G(f)$ are depicted on the right-hand side. Note that the frequency terms above 1 Hz and below -1 Hz have very small magnitudes.

5.B.5 The Fourier Transform

- The Fourier Transform of the complex exponential and trigonometric functions involve **Dirac's delta function**.
- Dirac's delta function is very useful in the study of all engineering fields including signals & systems. It is a pulse with unit area and infinitesimally short duration, i.e. an **impulse**:



$\epsilon \to 0$



Pulse of area one and width ϵ

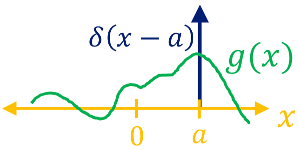
$$p(x) = \begin{cases} \frac{1}{\epsilon}, x \in (-\frac{\epsilon}{2}, \frac{\epsilon}{2}) \\ 0, \quad otherwise \end{cases}$$

Dirac's delta function

$$\delta(x) = \begin{cases} \infty, x = 0 \\ 0, \quad otherwise \end{cases}$$

- Properties of Dirac's delta function are:
  1) Because Area=1, $\int_{-\infty}^{\infty} \delta(x)dx = 1$.
  2) Given any function $g(x)$,

$$\int_{-\infty}^{\infty} g(x)\delta(x-a)dx = \int_{-\infty}^{\infty} g(a)\delta(x-a)dx = g(a)$$



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Not only energy signals have Fourier Transforms. For example, complex exponentials are power signals (see Section 4.A.1), and they do admit Fourier transforms. However, we need to slightly extend our mathematical vocabulary for this and introduce Dirac's delta function. This function has been very useful in almost all engineering fields including mechanical engineering, control engineering, radar engineering, telecommunications engineering etc.

Briefly, Dirac's delta function is an impulse: A pulse with an infinitesimally small duration and unit area. This can be viewed as the limit case of shrinking a pulse of with ϵ and height $1/\epsilon$ by taking ϵ towards zero. The resulting function is denoted by $\delta(x)$ and is zero everywhere except when its argument x takes the value zero. For $x = 0$, the function has to output infinity (i.e. ∞) to maintain the unit area. On the other hand, because ∞ is not a real number, $\delta(x)$ is not a real-valued function.

Nevertheless, it can be characterised by its properties. First, because Dirac's delta has an area of one, its integral over the real line is one. Second, the impulse can be shifted on the real line to an arbitrary position a by simply subtraction a from its argument. The integration of any signal with a shifted impulse equals to the function in the integrand evaluated at the impulse's location, i.e., $g(a)$.

Dirac's delta function will be very useful in defining the Fourier Transforms of the complex exponential and trigonometric functions.

For further visualisations, see https://www.youtube.com/watch?v=SxNVcCVj-3c

5.B.6 The Fourier Transform

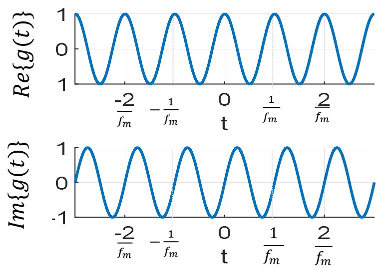
1. Impulse in the frequency domain $G(f) = \delta(f - f_m)$ is the Fourier Transform of the complex exponential $s(t) = e^{j2\pi f_m t}$ ((5) with frequency f_m):

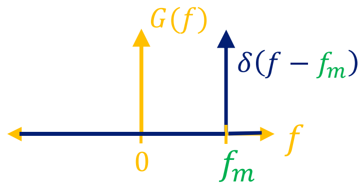
$$g(t) = e^{j2\pi f_m t} \quad \leftrightarrow \quad G(f) = \delta(f - f_m)$$

$$g(t) = \int_{-\infty}^{\infty} \delta(f - f_m) e^{j2\pi f t} df \quad \leftrightarrow \quad G(f) = \delta(f - f_m)$$

$$= e^{j2\pi f_m t}$$

Dirac's delta, property #2





The Fourier Transform of the complex exponential signal with frequency f_m is an impulse over the frequency axis shifted to f_m. In other words, $G(f) = \delta(f - f_m)$ and $s(t) = e^{j2\pi f_m t}$ form a Fourier Transform pair.

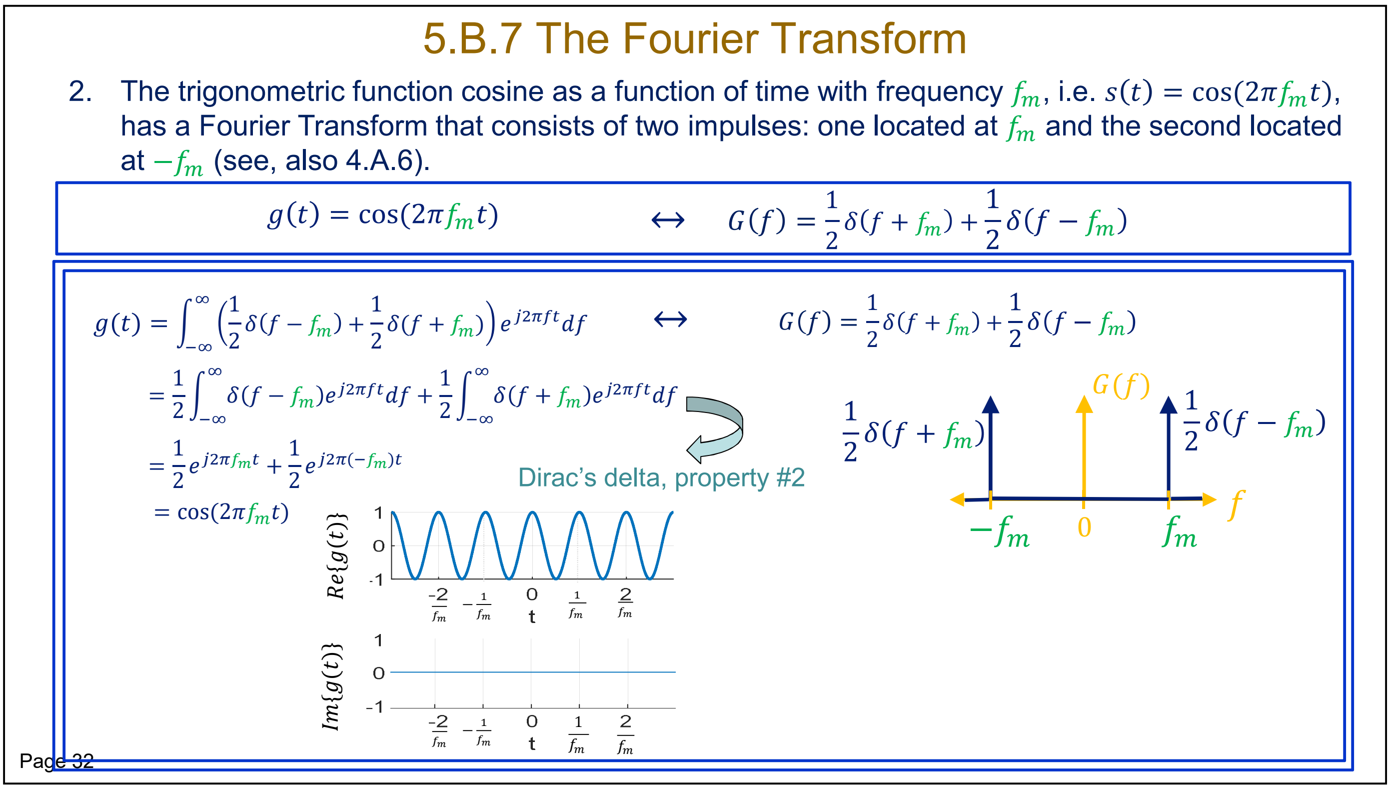
In order to verify this, let us find the signal that gets synthesised using the proposed spectrum in the Fourier Transform synthesis formula. Call this signal $g(t)$, then, using the second property of the Dirac delta function in Section 5.B.5

$g(t) = \int_{-\infty}^{\infty} \delta(f - f_m) e^{j2\pi f t} df$

$g(t) = e^{j2\pi f_m t}$

which is the complex exponential with frequency f_c proving that $G(f) = \delta(f - f_m)$ is its Fourier Transform.

Note that the evaluation of the Fourier Transform analysis formula for a complex exponential is not trivial. This is left out of the scope of this module.



The Fourier Transform of the trigonometric function $\cos(2\pi f_m t)$ consists of two impulse over the frequency axis shifted to f_m and $-f_m$. In other words, $G(f) = 0.5\ \delta(f - f_m) + 0.5\ \delta(f + f_m)$ and $\mathrm{s(t)} = \cos(2\pi f_m t)$ form a Fourier Transform pair.

This can be verified by simply substituting the proposed spectrum in the Fourier Transform synthesis formula and using the second property of the Dirac delta function in Section 5.B.5 as follows: let's denote the function that will be synthesised $g(t)$. Therefore,

$$g(t) = \int_{-\infty}^{\infty}(0.5\ \delta(f - f_m) + 0.5\ \delta(f + f_m))e^{j2\pi ft}df$$

$$g(t) = 0.5\int_{-\infty}^{\infty}\delta(f - f_m)e^{j2\pi ft}df + 0.5\int_{-\infty}^{\infty}\delta(f + f_m)e^{j2\pi ft}df$$

$$g(t) = 0.5e^{j2\pi f_m t} + 0.5e^{j2\pi(-f_m)t}$$

Now, use the Euler's formula (1) and expand the complex exponentials above:

$$g(t) = 0.5(\cos(2\pi f_m t) + j0.5sin(2\pi f_m t)) + 0.5(\cos(-2\pi f_m t) + j0.5sin(-2\pi f_m t))$$

The cosine is an even function and sine is an odd function (see Section 2.A.2)

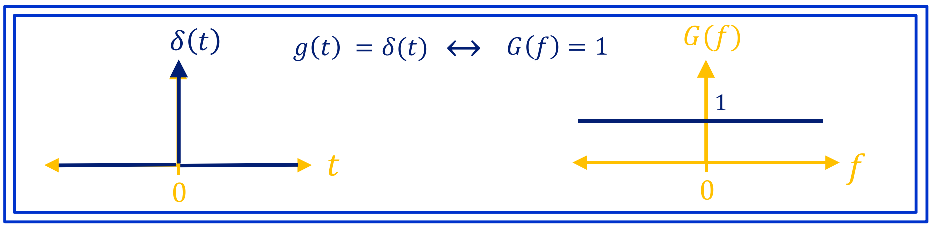
$$g(t) = 0.5\cos(2\pi f_m t) + 0.5\cos(2\pi f_m t) + \mathrm{j}0.5sin(2\pi f_m t) - j0.5sin(2\pi f_m t)$$

$$g(t) = \cos(2\pi f_m t).$$

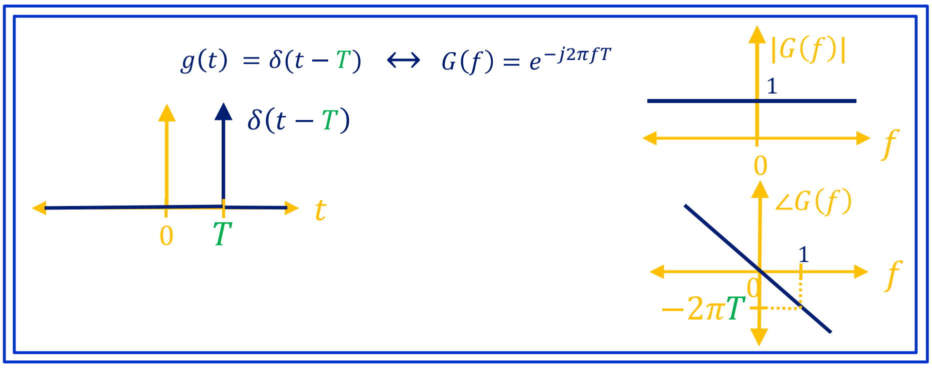
Similar to the complex exponential case, the evaluation of the Fourier Transform analysis formula for a sinusoidal signal is non-trivial and left out of the scope of this module.

5.B.8 The Fourier Transform

3. Impulse in the time-domain $g(t) = \delta(x)$ has an all-one "flat" spectrum:



A shift in the time-domain corresponds to a linear phase $\angle G(f)$:



An impulse signal in the time-domain $\delta(t)$ has a flat spectrum: In other words, if one adds up all complex exponentials of all frequency by weighting them with df, the end result is zero everywhere except the origin and tends to ∞ at the origin. This can be verified by simply substituting $\delta(t)$in the Fourier Transform analysis formula and using the second property of the Dirac delta function in Section 5.B.5 as follows:

$$G(f) = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi ft} dt,$$
$$= e^{-j2\pi ft}\Big|_{t=0},$$
$$= 1.$$

This spectrum is of unity magnitude and zero phase, i.e., the angle $\angle G(f)$ for all f is zero. If the impulse is shifted in time to T, the corresponding spectrum still is flat in its magnitude, but linear in its phase:

$$G(f) = \int_{-\infty}^{\infty} \delta(t-T) e^{-j2\pi ft} dt\,,$$
$$= e^{-j2\pi ft}\Big|_{t=T}\,,$$
$$= e^{-j2\pi fT}.$$

Therefore, the absolute value of $G(f)$ is still unity, but the phase $\angle G(f)$ is now linear with f. In particular, the slope of the linear phase is determined by the time shift as $-2\pi T$.

6.A.1 Systems: Definition

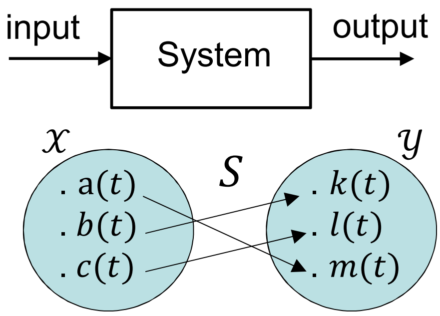
- Dictionary definition

  system (n) *a construct or collection of different elements that together produce results not obtainable by the elements alone. The elements, or parts, can include people, hardware, software, facilities, policies, and documents; that is, all things required to produce systems-level results.*

- Analytical (mathematical) definition

  system (n) *a function* that maps an input signal to an output signal.

- A system produces an output signal in response to an input signal.
- On the right-hand side is a system $S: \mathcal{X} \rightarrow \mathcal{Y}$ whose input-output pairs are $S = \{(a(t), m(t)), (b(t), k(t)), (c(t), k(t))\}$.
- For example, in response to the input $a(t)$, the system S outputs $m(t)$ which is written $m = S(a)$.
- More complex systems are built by interconnecting systems.



The dictionary definition of a system on the slide refers to interaction of components each of which perform a certain process to transform inputs to the process to the outputs. In this module, we are interested in an analytical definition: A system is a function which maps input signals to output signals. In other words, a system is a transformation that outputs a specific signal in response to a specific input. For example, on the bottom-right is a Venn-diagram representation of a system S which maps the signal set $\mathcal{X}$ onto another signal set $\mathcal{Y}$, i.e. $S: \mathcal{X} \rightarrow \mathcal{Y}$ (see, also Section 1.B.1 for comparison).

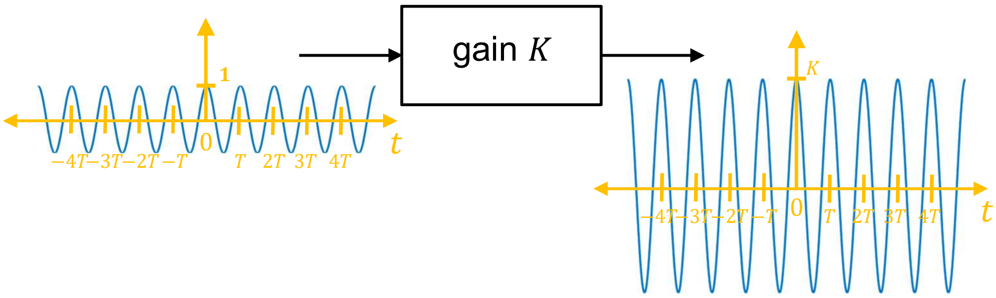
One might list the input-output signal pairs to represent the system S as a list. Alternatively, the output to a specific signal, say $a(t)$ can be written using the regular function notation $k = S(a)$, which reads "k equals to S of a." Note that S is NOT a composite function; it maps the signal a in the set of signals $\mathcal{X}$ to another function k which is an element of $\mathcal{Y}$. Both signals a and k maps time value t to a real or complex number.

More complex systems can be viewed as interconnections of such systems. For example, an aircraft control system is an interconnection of several systems [2]; the aircraft as a system is described by its equations of motion and aerodynamic forces mapping control surfaces, engine thrust and other inputs to rotation rates and other relevant kinematic quantities. This system is connected to the sensors, which measure various quantities including the heading and accelerations. An autopilot is a system which maps commands from the pilot such as the desired course, altitude, and speed, and the sensor measurements to actuators associated with the control surfaces (e.g. rudder, tail and ailerons).

6.B.1 Examples

1. An **amplifier** multiplies the input signal with a gain factor much larger than 1, i.e. $K \gg 1$).

Example 6: Suppose $g(t) = \cos(2\pi \frac{1}{T} t)$, then, the output is $K \cos\left(2\pi \frac{1}{T} t\right)$:

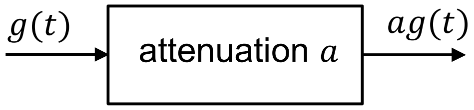


Let us start with a simple example and consider an amplifier system: the input signal is multiplied with a gain factor of K which is much larger than 1 (for example, K might equal to 10 or 100). This is denoted by $\gg$ and reads “much larger than”. One can show using the power formula in Section 3 that multiplication by K results by increasing the energy or power of a signal by K^2.

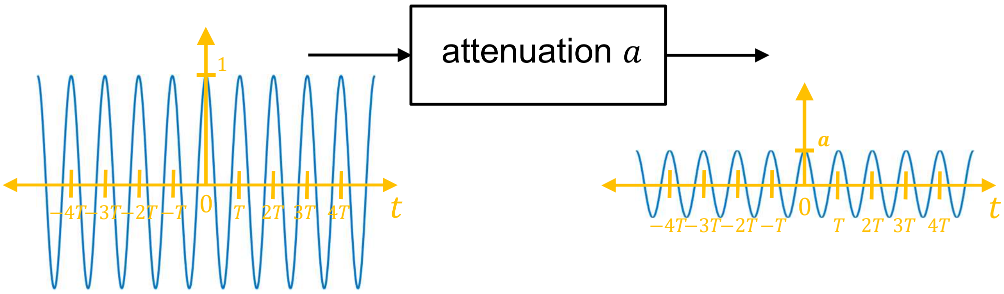
As an example, let us consider a cosine signal at the input that has a period of T. The graph of this signal is given on the left hand side. The system transform it to a cosine with the same period (and frequency $f = 1/T$), but higher amplitude. The amplitude of 1 at the input means the output cosine’s amplitude is K.

6.B.2 Examples

2. An **attenuator** multiplies the input signal with a gain factor that is a small number between $0 < a < 1$:



Example 7: Suppose $g(t) = \cos(2\pi \frac{1}{T} t)$, then, the output is $a\cos(2\pi \frac{1}{T} t)$:

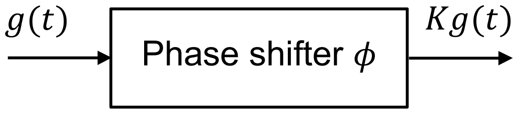


Now consider an attenuator system: the input is mapped to a version of itself that is multiplied with a number less than one (but greater than zero). Here, the attenuation factor a might be 0.1 or 0.01. Note that because $0 < a < 1$, $0 < a^2 < a$ and the output power/energy signal will be the input with much less power/energy (see, Section 3). For example, if $a = 0.1$, the output will be the input signal with $1/100$ of its power/energy.

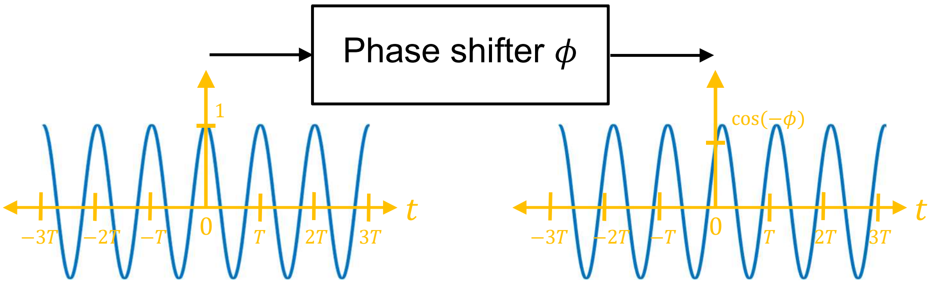
If a cosine is input to the attenuator, the output will be a cosine with the same period T (and frequency $f = 1/T$). The amplitude at the output, however, will be α.

6.B.3 Examples

3. A **phase shifter** adds phase to the input signal:



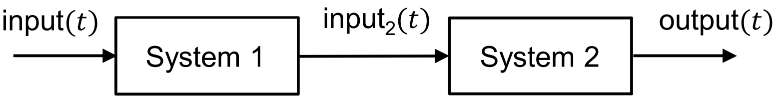
Example 8: Suppose $g(t) = \cos(2\pi\frac{1}{T}t)$, then, the output is $\cos(2\pi\frac{1}{T}t - \phi)$:



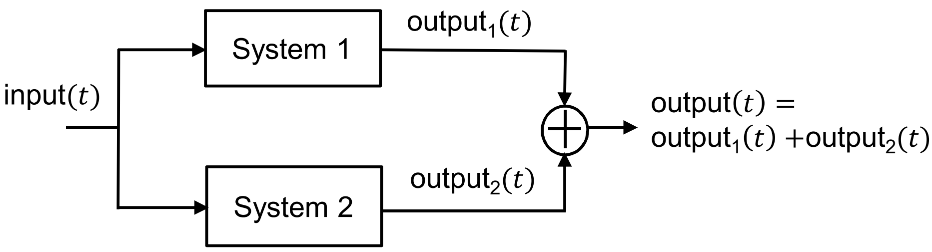
In this example, we consider a phase shifter system: The input's phase is shifted by ϕ radians to produce the output. For example, if the input is a cosine of period T (and, frequency $f = 1/T$), the output will be a cosine of the same period (and, frequency) but with phase ϕ. Therefore, the input signal with zero phase is 1 at $t = 0$ and the output is $\cos(-\phi)$ at $t = 0$.

6.C.1 Interconnections

1. A **series (cascade)** interconnection

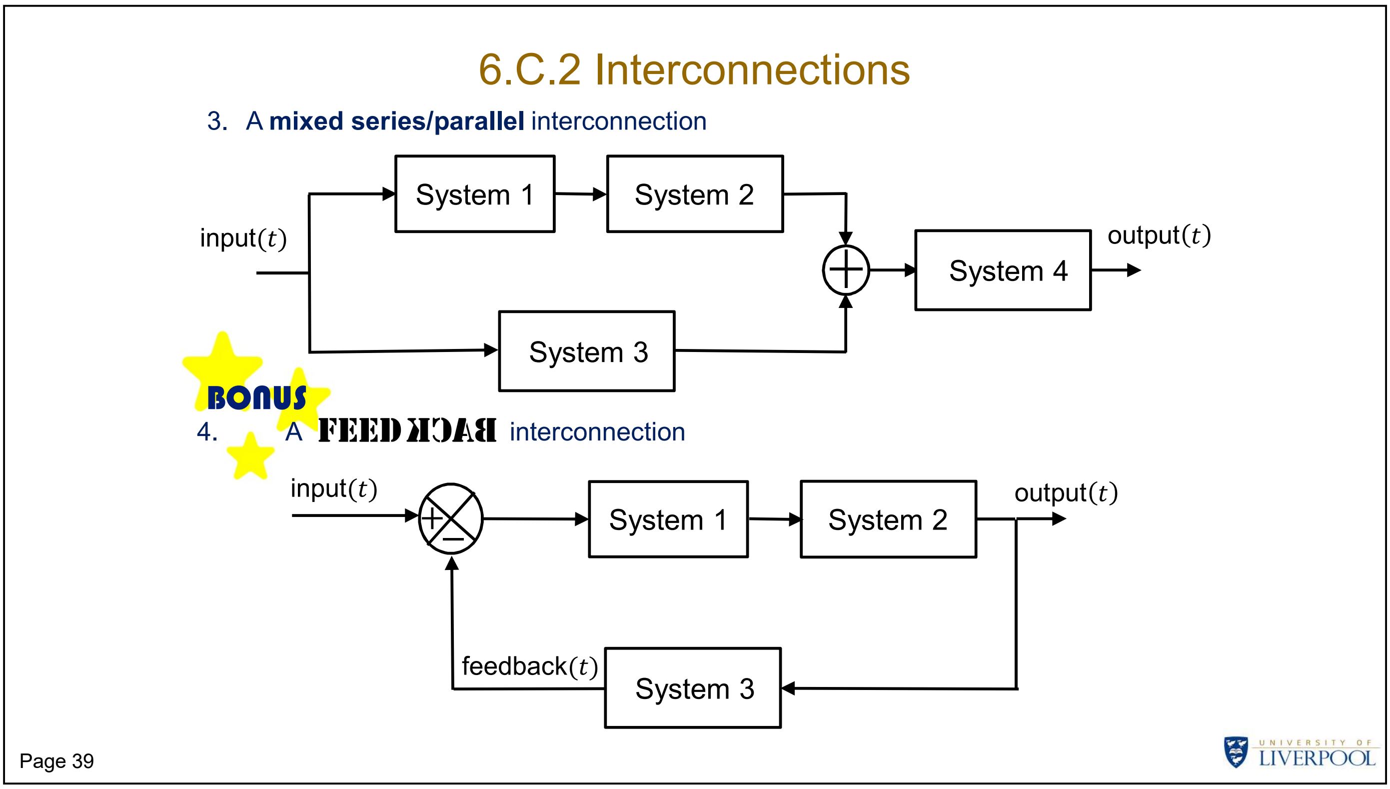


2. A **parallel** interconnection



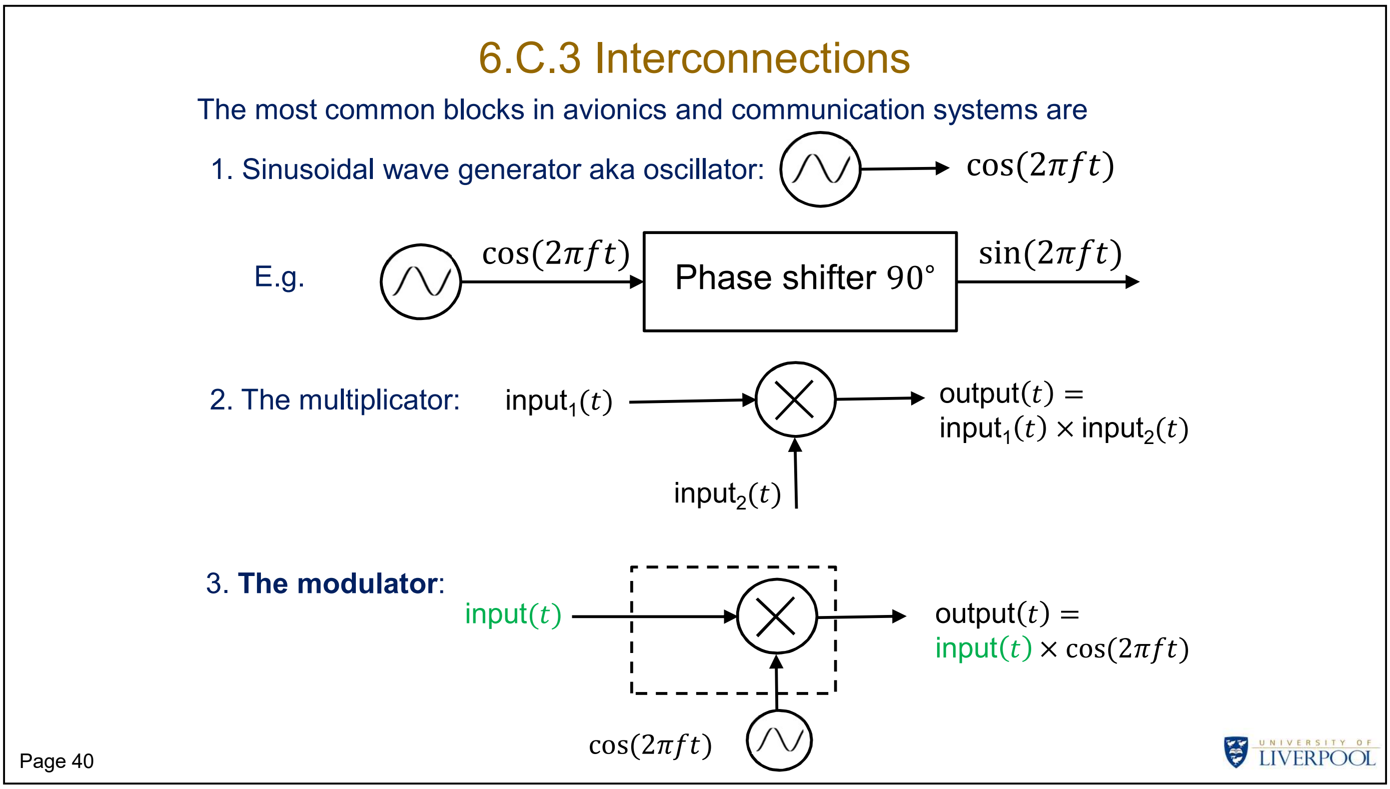
Many systems are interconnections of simpler systems as explained in Section 6.A.1. The first interconnection type is a series (or cascade) interconnection: the input to the cascade is transformed to an output by the first system on the left, which then becomes the input to the second system (denoted by input2). The output of the overall cascade is the response of the second system to the output of the first one (i.e. input2) for the selected input.

The second fundamental interconnection type is the parallel connection. In this case, the input to the parallel systems is the same. The output of the interconnection is the summation of all of the outputs of the parallelly connected systems.



More complex systems are built by using a mix of series and parallel connections. An example is on the top of slide: there is a parallel interconnection the first branch of which consists of two systems in series, i.e. system 1 and system 2. The second branch of the parallel connection is system 3. The parallel interconnection is in series to system 4. Hence, system 4 produces the final output in response to the summation of the outputs from system 2 and 3. System 3 produces its output in response to input(t). System 2 produces its output in response to the output of system 1 which in turn maps input(t) to its output.

The last interconnection example on the slide is the "feedback" interconnection, which is commonly used in control systems such as the fly-by-wire and autopilot. Here, the final output of the interconnection is connected back to the input via a feedback path. This allows control systems to compare the output and the input of the system; the input is often the desired outcome which can be, for example, the desired altitude or the heading angle etc. The difference of the input and the feedback is the error the overall output has with respect to the desired input. System 1 converts this error signal to a control signal to System 2. System 2 represents the system that is controlled and produces an output in response to the control system. System 3 converts this output to a compatible signal to the input, for comparison. This loop continues in every "split second" or, mathematically, with infinitesimal time steps. Note that if the feedback equals to the desired input, then there will be zero input to the System 1 indicating that no change in the overall output of the system is required.



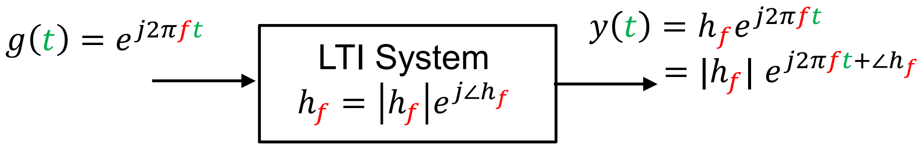
There are three blocks which are very common in the study of communication systems and avionics. The first is a "source" block and called the sinusoidal wave generator or an oscillator. The output of this block is a cosine signal at frequency f. Sometimes, a sine wave is needed; a phase shifter in serial connection to a cosine oscillator outputs a sine wave because $\sin(x) = \cos(x - \pi/2)$ (see, Section 2.A.2).

The second common block is the multiplicator –which complements the addition block in parallel connections. The output of a multiplicator is the product of its inputs which on the slide are input1(t) and input2(t).

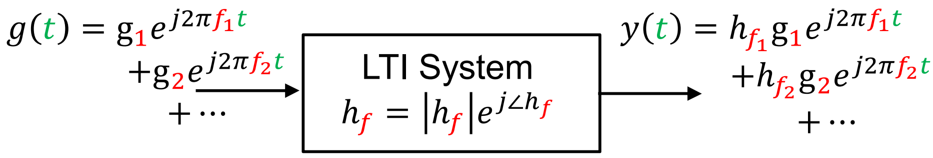
One of the most important blocks in communication systems and avionics is the modulator: This system multiplies its input with a cosine signal, i.e. the output is the product of the input with a cosine at frequency f. Note that some modulators might use a sine signal. In this module, the trigonometric function related to a modulator will be stated explicitly.

7.1 Linear Time-Invariant Systems

- An important class of systems in avionics and communication systems is the **linear *and* time-invariant (LTI) systems.**
- **(Narrative Jump)**
  - Why **LTI** systems are important?
  - Because the response of an **LTI** system to any complex exponential is a complex exponential at the same frequency weighted with a **system specific** complex number.



- Moreover, the output to a **sum** of weighted complex exponentials is a superposition of same terms scaled with system specific complex numbers.



Systems come with an unlimited variety of ways to map input signals to output signals. Therefore, it is useful to classify them to identify the ones that exhibit useful patterns of behaviour.

One such useful class consists of systems which are both linear and time-invariant.

Let us make a narrative jump and before introducing what makes a system linear or time-invariant, let us state what makes them important, first.

The first important property of LTI systems is that their response to a complex exponential (see, Section 4.A.1) is a weighted version of the same complex exponential with a complex number. Therefore, the output will be a complex coefficient with the same frequency. The magnitude and the phase of the output will be specified by the complex weight the system uses to transform the input. On the slide, a complex exponential of frequency f is input to an LTI system which weights it with a complex number h_f. Therefore, the output has a magnitude of $|h_f|$ and phase of $\angle h_f$. The subscript f means that this coefficient is frequency dependant, i.e. for two different frequencies f_1 and f_2, the system will multiply the complex exponentials by h_{f_1} and h_{f_2}, respectively.

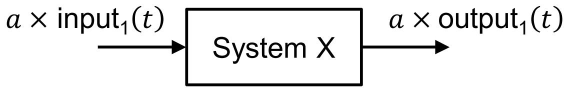
The second important property of LTI systems is that their response to a sum of weighted complex exponentials will be the sum of responses to individual components of the input. This property will be very useful in characterising LTI systems and establishing their input/output mapping rules because all signals can be written as a sum of complex exponentials as revealed by our discussion on Fourier series and the Fourier transform.

7.2 Linear Time-Invariant Systems

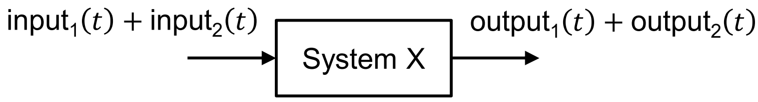
- What is a **(1) linear** *and* **(2) time-invariant** system?
- Consider two input/output pairs for an otherwise unknown system X:

1. **Linear Systems**: System X is **linear** if the following two conditions are satisfied for all arbitrary selection of input$_1(t)$ and input$_2(t)$

   I. The response to $a \times$ input$_1(t)$ is $a \times$ output$_1(t)$ for any complex number $a \in \mathbb{C}$.



   II. The response to input$_1(t)$ + input$_2(t)$ is output$_1(t)$ + output$_2(t)$



Suppose that we have an unknown system called System X, and we know that it produces output1(t) in response to input1(t) and output2(t) in response to input2(t). There are two conditions to test to see if System X is linear:

Choose a complex number $a \in \mathbb{C}$ and multiply the input with this number to have a scaled input signal. If the output scales exactly with a, i.e. System X yields $a \times$ output1(t) in response to $a \times$ input1(t), and does so for every possible a, then System X passes the first test of linearity.

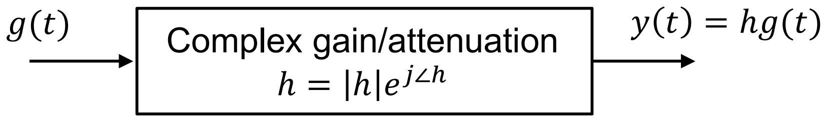
Now add the two test inputs input1(t) and input2(t) together to drive the system. If the output is the summation of output1(t) and output2(t), System X passes the second test of linearity.

A system is linear if the above two conditions hold.

Note that these conditions ascertain that the output of System X can be found from test input/output pairs when the input signal in questions is a weighted superposition of the test inputs. For example, consider the weighted superposition examples on Slide 4.A.3 and suppose that we know that System X responds to the complex exponentials of frequency $f, 2f$ and $3f$ as $output_1(t), output_2(t)$, and $output_3(t)$, respectively. If System X is linear, i.e. the above two conditions hold, then the output of System X to $g_2(t)$ on Slide 4.A.3 is going to be $\alpha_1 output_1(t) + \alpha_2 output_2(t)$. This means that for any different input $g_2(t)$ one can obtain by varying the weights α_1 and α_2, we can find the output by substituting these values in $\alpha_1 output_1(t) + \alpha_2 output_2(t)$. Similarly, we can find the output in response to all different input $g_3(t)$ that one can obtain by varying three weights α_1, α_2 and α_3 by substituting them in $\alpha_1 output_1(t) + \alpha_2 output_2(t) + \alpha_3 output_3(t)$.

7.3 Linear Time-Invariant Systems

Example 9: The complex gain/attenuation is a linear system.



The complex gain/attenuation response to $g_1(t)$ is $y_1(t) = hg_1(t)$ and the response to $g_2(t)$ is $y_2(t) = hg_2(t)$.

I. The response to $ag_1(t)$ is $hag_1(t) = ahg_1(t)$ which equals to $= ay_1(t)$.
Therefore, the first condition is satisfied.

II. The response to $g_1(t) + g_2(t)$ is $h\big(g_1(t) + g_2(t)\big) = hg_1(t) + hg_2(t)$ which equals to $= y_1(t) + y_2(t)$.
Therefore, the second condition is satisfied.

As an example, let's investigate the linearity of the complex gain/attenuation system which simply multiplies its input with a non-zero complex number h. Because h is a complex number, it has a magnitude denoted by $|h|$ and a phase denoted by $\angle h$(see, slide 2.B.2). When multiplied with h, the magnitude of the input will be weighted by $|h|$ and its phase will be added $\angle h$. If the magnitude is larger than one, the system will have an amplification effect, if the magnitude is smaller than one, the system will have an attenuation effect.

The output in response to an arbitrary signal $g_1(t)$ is $y_1(t) = hg_1(t)$. Similarly, for another input $g_2(t)$ the output will be $y_2(t) = hg_2(t)$.

Now, let's test the conditions of linearity:

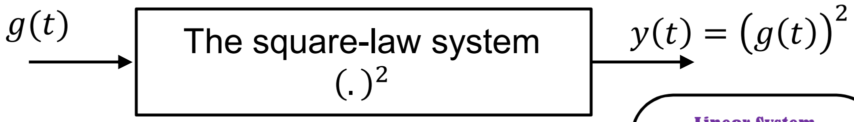
If we scale the first input by a and drive the system with $ag_1(t)$, the output will be $h\left(a\, g_1(t)\right)$, which is nothing but h times the input. This quantity is also a times the output to $g_1(t)$ as a result of the commutativity of multiplication:
$h\left(a\, g_1(t)\right) = a(h\, g_1(t))$ and thus $h\left(a\, g_1(t)\right) = a\, y_1(t)$.
As a result, the first condition of linearity is satisfied.

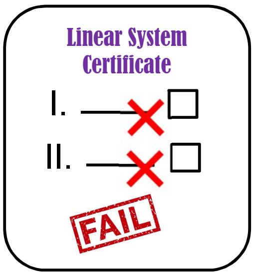
If we drive the system with the summation of $g_1(t)$ and $g_2(t)$, the output is $h(\, g_1(t) + g_2(t))$ and in turn, as a result of the distributive property of multiplication, $h\big(\, g_1(t) + g_2(t)\big) = hg_1(t) + hg_2(t)$. This quantity is nothing but the individual outputs, i.e., $h\big(\, g_1(t) + g_2(t)\big) = y_1(t) + y_2(t)$. Therefore, the second condition of linearity is also satisfied.

As a result, the complex gain/attenuation system is a linear system.

7.4 Linear Time-Invariant Systems

Example 10: The square-law is **not** a linear system – it is a **non-linear** system.





The square-law output to $g_1(t)$ is $y_1(t) = \left(g_1(t)\right)^2$ and the response to $g_2(t)$ is $y_2(t) = \left(g_2(t)\right)^2$.

I. The response to $a g_1(t)$ is $\left(a g_1(t)\right)^2 = a^2\left(g_1(t)\right)^2$ which equals to $= a^2 y_1(t)$.
The first condition is NOT satisfied (should have been $a y_1(t)$).

II. The response to $g_1(t) + g_2(t)$ is $\left(g_1(t) + g_2(t)\right)^2 = \left(g_1(t)\right)^2 + \left(g_2(t)\right)^2 + 2 g_1(t) g_2(t)$ which equals to $= y_1(t) + y_2(t) + 2 g_1(t) g_2(t)$.
Therefore, the second condition is NOT satisfied (should have been only $y_1(t) + y_2(t)$).

As a counter example, let's investigate the linearity of a system which maps inputs to outputs using a square-law; it finds the square at the input as its output. For example, its output in response to an arbitrary signal $g_1(t)$ is $y_1(t) = \left(g_1(t)\right)^2$. Similarly, for another input $g_2(t)$ the output will be $y_2(t) = \left(g_2(t)\right)^2$.

Now, let's test the conditions of linearity:

If we scale the first input by a and drive the system with $a g_1(t)$, the output will be $y_1(t) = \left(a\, g_1(t)\right)^2$. This quantity is also a^2 times the output to $g_1(t)$:
$\left(a\, g_1(t)\right)^2 = a^2\left(g_1(t)\right)^2$ and thus $\left(a\, g_1(t)\right)^2 = a^2\, y_1(t)$.
As a result, the first condition of linearity is not satisfied. This output should have been $a y_1(t)$ to satisfy the condition.

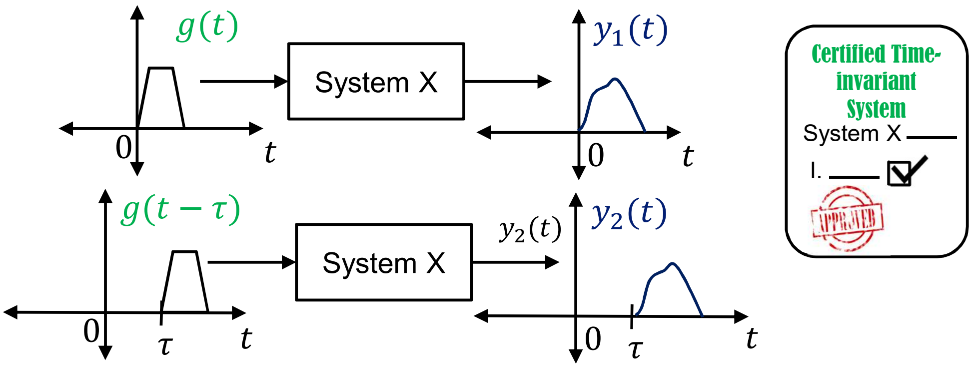
If we drive the system with the summation of $g_1(t)$ and $g_2(t)$, the output is $\left(g_1(t) + g_2(t)\right)^2$. After expanding the sum of squares, we find that this quantity equals to $\left(g_1(t)\right)^2 + \left(g_2(t)\right)^2 + 2 g_1(t) g_2(t)$, which is nothing but the individual outputs added twice the product of the inputs, i.e., $y_1(t) + y_2(t) + 2 g_1(t) g_2(t)$. Therefore, the second condition of linearity is NOT satisfied, either. The output should have been $y_1(t) + y_2(t)$ to satisfy the condition.

As a result, the square-law is not a linear system. Such systems are referred to as non-linear systems.

7.5 Linear Time-Invariant Systems

- **Time-invariance** is about maintaining the behaviour and characteristics unchanged over time.

2. **Time-invariant Systems**: A system is **time-invariant** if the response to a time-shifted input is the response to the original signal shifted in time with the same amount.
   In other words, if an arbitrary input signal $g(t)$ induces $y_1(t)$ and its τ shifted version $g(t-\tau)$ induces $y_2(t)$, the system is time-invariant if $y_2(t) = y_1(t-\tau)$.



The second important property of a system is related to whether it maintains its behaviour and characteristic unchanged over time. Systems which remain unchanged in this aspect are called time-invariant.

A more precise descriptor for a time-invariant system makes use of time-shifted versions of input signals. In particular, if a time-shifted input induces the same output as the original signal with a time-shift that equals to the time-shift of the input, then the system is time-invariant.

Mathematically, a time-shift of τ is introduced by subtracting τ from the time-variable t, e.g. $g(t-\tau)$ is the τ shifted version of $g(t)$ in time. This can be verified by noticing that

$g(t-\tau)|_{t=\tau} = g(t)|_{t=0}$ (reads g of t minus τ evaluated at t equals to τ equals to g of t evaluated at t equals to 0),

i.e. the origin point $t = 0$ of $g(t)$ is at $t = \tau$ for $g(t-\tau)$.

An unknown system (referred to as System X here) is time-invariant if the output in response to $g(t)$ denoted by $y_1(t)$ becomes shifted in time by τ when the input is replaced by $g(t-\tau)$.

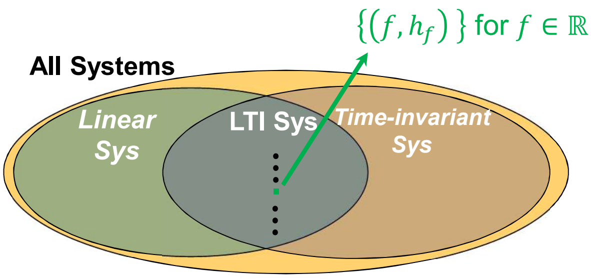
The linear system in Example 9 and the non-linear system in Example 10 are both time-invariant systems (prove as an exercise).

8.1 Characterisation of LTI Systems: Frequency response

- If a system is **linear** and **time-invariant (LTI)**, then its response to any complex exponential is a weighted complex exponential with the same frequency. The weight is assigned to that particular frequency by the system.

$$s(t) = e^{j2\pi f t} \longrightarrow \boxed{\text{LTI System} \quad h_f = |h_f| e^{j\angle h_f}} \longrightarrow w(t) = h_f e^{j2\pi f t}$$

- An LTI system thus can be represented by the list of pairs $\{(f, h_f)\}$ for all $f \in \mathbb{R}$.



- The function $H: \mathbb{R} \to \mathbb{C}$ that maps frequency value $f \in \mathbb{R}$ to the coefficient h_f is called **the frequency response** of the system, i.e.

$$\boxed{H(f) = h_f}$$

is the system's frequency response.

The output of an LTI system in response to a complex exponential is a weighted version of the complex exponential, where the weight is a complex number h_f (i.e. the weight consists of a magnitude $|h_f|$

and a phase $\angle h_f$), and depends on the frequency. In other words, an LTI system has a complex gain h_f reserved for a complex exponential of frequency f. These weights uniquely characterise LTI systems; two systems are identical only when their corresponding weights are the same for all different frequencies $f \in \mathbb{R}$.

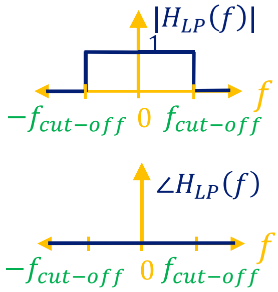
One can capture all such weights in a function that maps frequencies to complex weights, i.e. define a function $H: \mathbb{R} \to \mathbb{C}$ that maps frequency value $f \in \mathbb{R}$ to the coefficient h_f. This function is called the frequency response of the system. LTI systems are characterised uniquely by their frequency response.

In other words, if we consider the set of all systems and the subset of LTI systems (which is at the intersection of the subset of linear systems and the subset of time-invariant systems), each element in the subset of LTI system can be identified by its frequency response.

8.2 Characterisation of LTI Systems: Examples

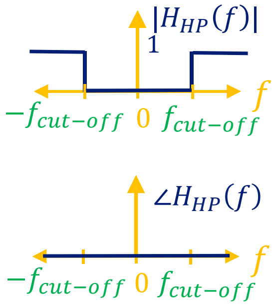
- Example 11: An ideal **low pass filter** multiplies complex exponentials with low frequencies within $-f_{cut-off} < f < f_{cut-off}$ with one, and others with zero. Here, $f_{cut-off}$ is called the cut-off frequency. Therefore, the (ideal) low pass frequency response is

$$H_{LP}(f) = \begin{cases} 1, & |f| < f_{cut-o} \quad , \\ 0, & |f| \geq f_{cut-o} \quad . \end{cases}$$



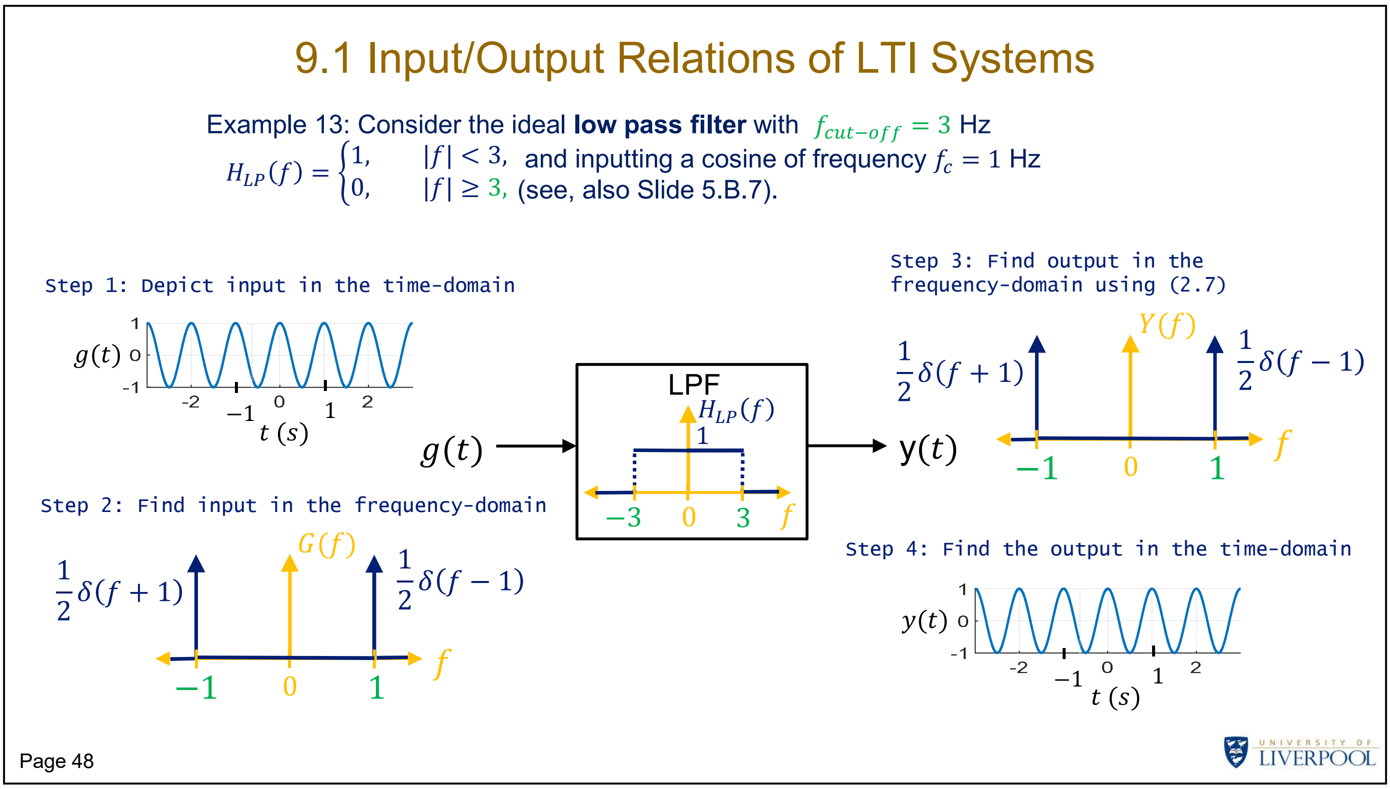
- Example 12: An ideal **high pass filter** multiplies complex exponentials with high frequencies $|f| > f_{cut-off}$ with one, and others with zero. Also here $f_{cut-off}$ is called the cut-off frequency. Therefore, the (ideal) high pass frequency response is

$$H_{HP}(f) = \begin{cases} 1, & |f| > f_{cut-off}, \\ 0, & |f| \leq f_{cut-off}. \end{cases}$$



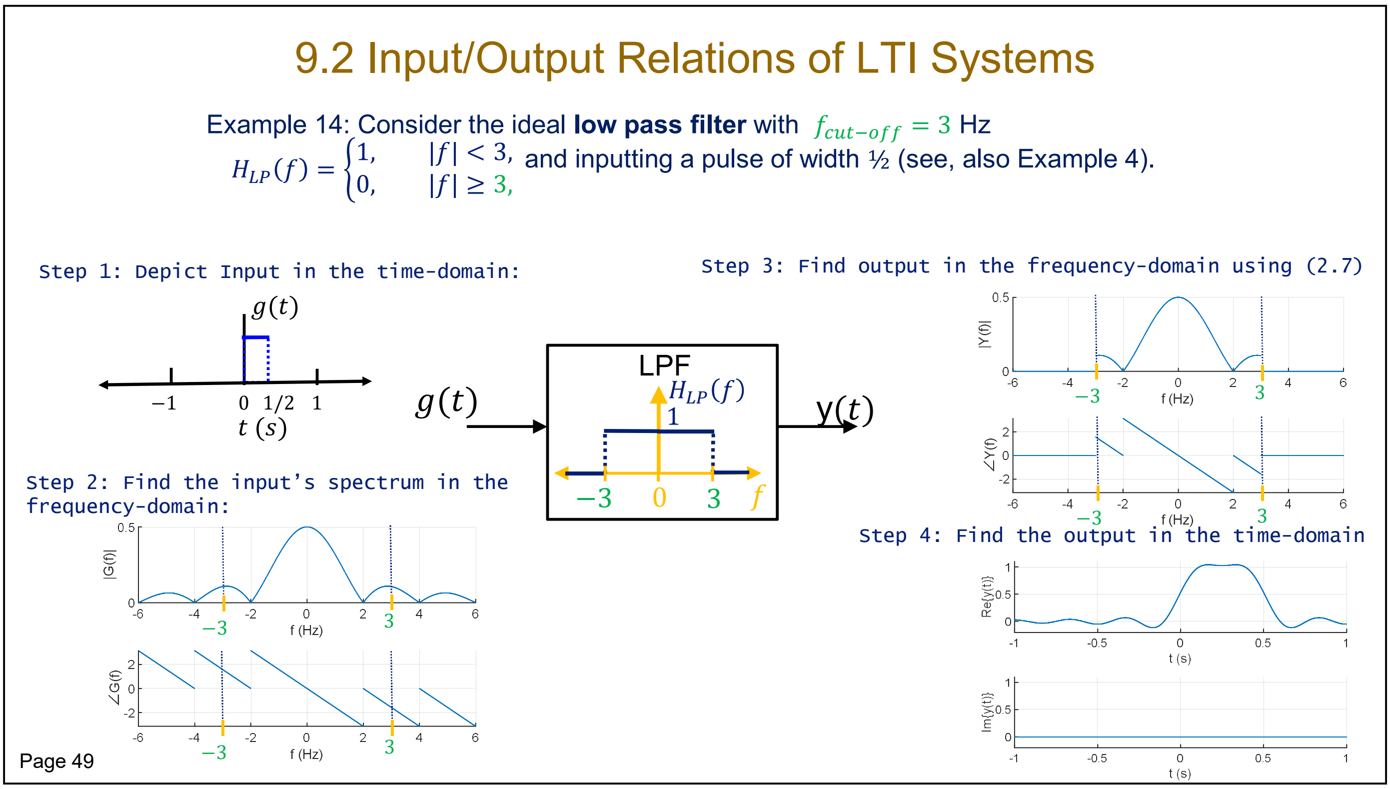
As an example, let us consider the ideal low-pass filter. This system is an LTI system with a frequency response that is of magnitude 1 (and phase zero) for frequencies f between $-f_{cut-off} < f < f_{cut-off}$ and zero for all other frequencies. Here, the cut-off frequency $f_{cut-off}$ is a user selected parameter. The output of this system is zero if the input is a complex exponential at a frequency higher than the selected cut-off frequency $f_{cut-off}$ or lower than $-f_{cut-off}$. The characterising frequency response in terms of its magnitude and phase are depicted on the top right figures on the slide.

A complementary example is an ideal high-pass filter. This system is also an LTI system with a frequency response that is of magnitude 1 (and phase zero) for frequencies f that is higher than a selected cut-off, i.e., $f > f_{cut-off}$ or lower than the negative cut-off frequency, i.e. $f < -f_{cut-off}$. The characterising frequency response are given on the bottom right figures on the slide.



The frequency response of $H(f)$ of an LTI system is very useful in finding its output in response to any arbitrary input: for example, consider low-pass filtering a rectangular pulse of width ½. The time-domain representation of the input signal is depicted on the top-left figure on the slide. The spectrum of this input revealing its complex exponential components (see, Section 5.B) is on the bottom-left found using the Fourier Transform analysis formula (10).

In this example, the cut-off frequency is selected as 3 Hz. The output of the LPF is going to supress all frequencies that are not between -3 Hz and 3 Hz. Therefore, the output spectrum has to be the spectrum depicted on the bottom right. In other words, the frequency-domain representation of the output equals to the input spectrum for all frequencies f between $-3 < f < 3$ and zero, otherwise. The output we seek to find is therefore the time-domain representation of this spectrum, which can be found using the Fourier Transform synthesis formula (11). The time-domain representation of the output signal has thus the graph on the top-right.



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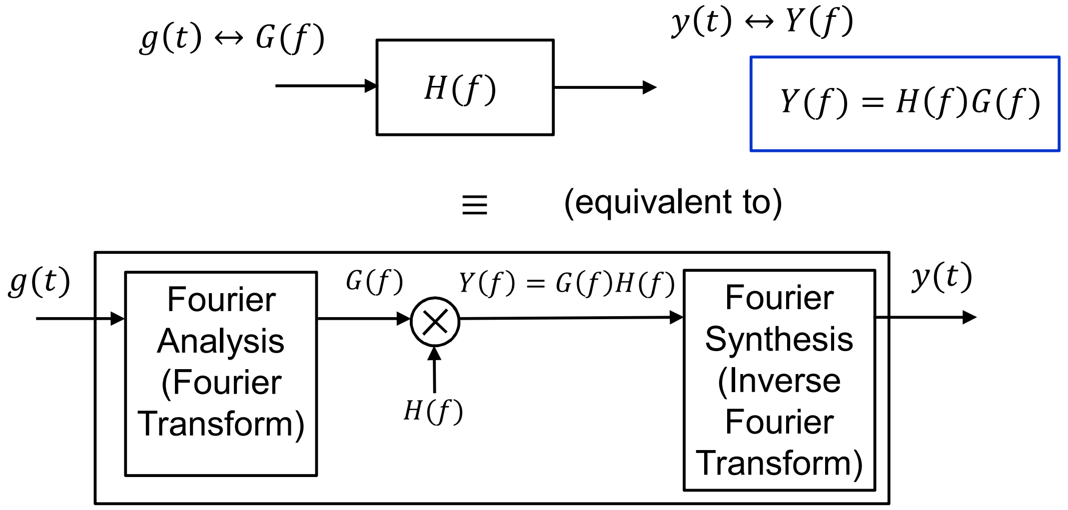
9.3 Input/Output Relations of LTI Systems

- The **input/output relation** of an **LTI** system with a frequency response $H(f)$ is thus: The spectrum of the output signal is the multiplication of the input spectrum with the system's frequency response.



The previous example can be generalised to find the output $y(t)$ of any LTI system that is characterised by its frequency response $H(f)$ in response to an arbitrary input signal $g(t)$. As established in Section 5, the time-domain and frequency-domain representations (or spectrum, see, Section 5.B.4) of a signal are equivalent, i.e., they are two sides of the same coin. The output of an LTI system to $g(t)$ will hence have a spectrum that is the product of the input spectrum $G(f)$ and the system response $H(f)$. The time-domain representation of the output can be found by using the Fourier Transform synthesis formula with the output spectrum $Y(f) = G(f)H(f)$.

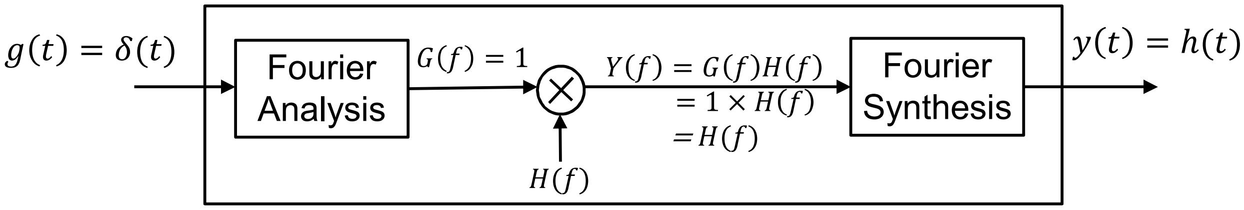
Here, the Fourier Analysis decomposes the input signal into as an infinite/finite sum of weighted complex exponentials and records those weights in the spectrum $G(f)$ (see, Equation (10)). In response to a complex exponential, an LTI system produces a version of it further weighted by $H(f)$ and in response to an infinite/finite sum of complex exponentials, the output will be their sum with each one multiplied by $H(f)$ corresponding to its frequency f. As the weight of the complex exponential of frequency f in the input is $G(f)$, its weight at the output becomes $G(f)H(f)$. Therefore, the spectrum of the output signal is $Y(f) = G(f)H(f)$. The time-domain representation of the output can then be found simply by using the Fourier synthesis formula (see, Equation (11)) which finds the sum of complex exponentials weighted with $Y(f)$.

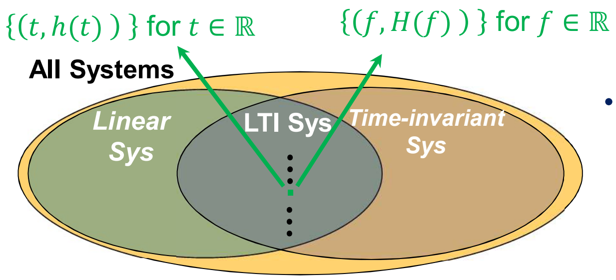
As a result, the frequency response of an LTI system specifies its input-output relation as described above by establishing which output signal it would generate in response to an arbitrary input. The result is a straightforward multiplication in the Frequency-domain demonstrating the power of using frequency domain representations.

9.3 Input/Output Relations of LTI Systems

BONUS

- An **LTI** system is also characterised by its **impulse response**, i.e. its output in response to an impulse:
  Remember that the time-domain impulse is one for all frequencies (see, Part I, Signals, Slide 3.C.3).





- The **frequency response** of the system, i.e. $H(f)$ and its **impulse response** $h(t)$ are Fourier Transform pairs.

The response of an LTI system to an impulse $\delta(t)$ at the input is called its impulse response. The spectrum of an impulse in time equals to one for all frequencies (see, Section 5.B.8). Therefore, the time-domain representation of an LTI system's response to an impulse is nothing but the inverse Fourier Transform of its frequency response. In other words, the frequency response of an LTI system and its impulse response are Fourier transform pairs.

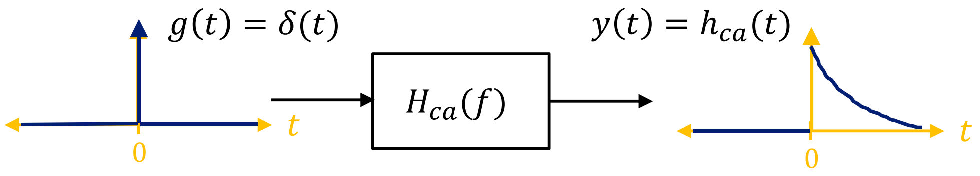
Thus, the impulse response provides a characterisation for an LTI system that is equivalent to a frequency response.

9.4 Input/Output Relations of LTI Systems

BONUS

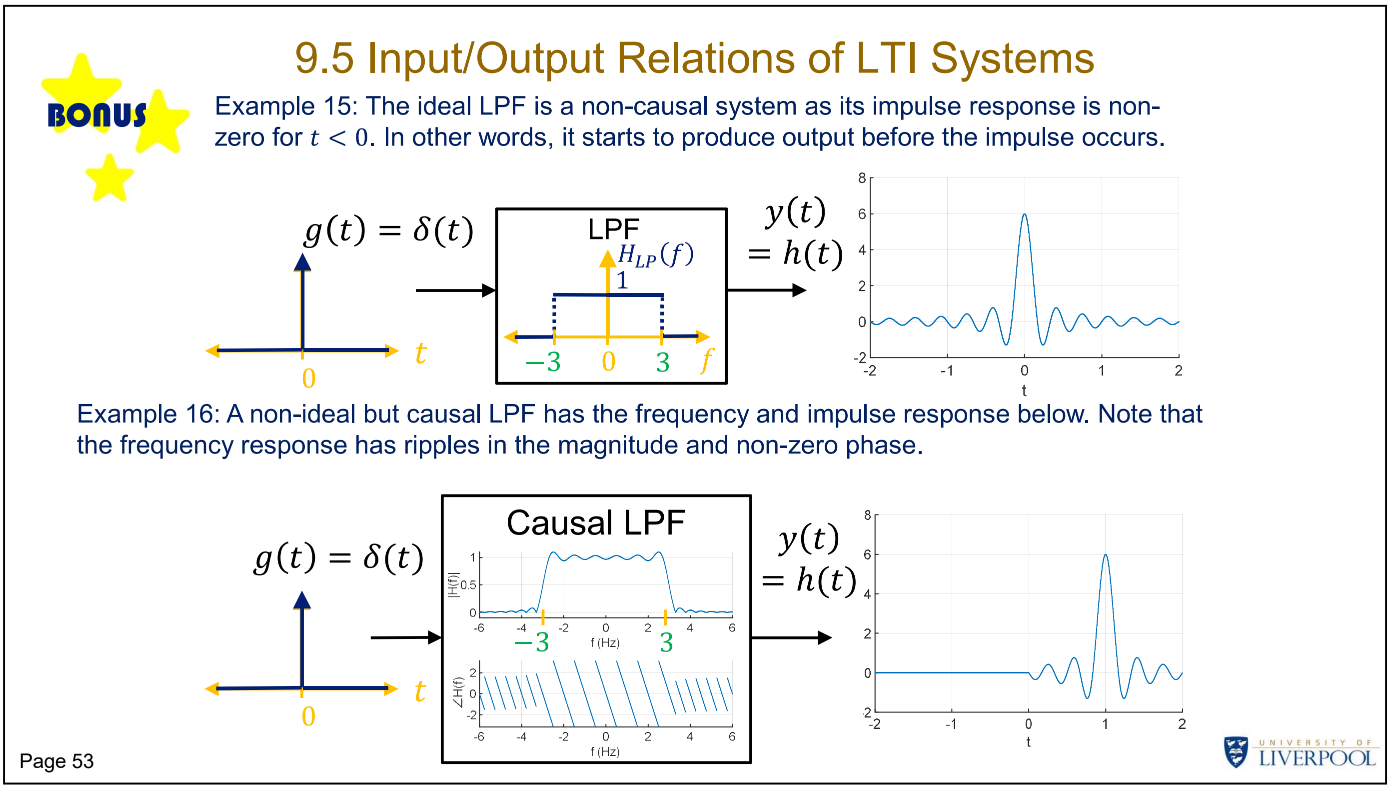
1. **The impulse response** reveals **causality properties of an LTI** system:
   LTI systems can be **causal** or **non-causal**. Causal systems are physically realisable in contrast to non-causal systems which cannot be realised.
   An LTI system is causal if its response to an impulse is zero for all time instants $t < 0$ (i.e. before the impulse occurred). Otherwise, it is non-causal

Example 14: The following system is causal as its impulse response is non-zero only for $t \geq 0$.



The impulse response $h(t)$ is a time-domain characterisation of an LTI system, and reveals an important property about the system; is this system physically realisable? Physically realisable systems are called causal systems and have an impulse response that is zero for all time instants before zero, i.e. $t < 0$. This means that the system does not produce any output before the impulse occurs at the time origin $t = 0$. Any non-zero value at the output thus comes at or after the impulse, i.e. for $t \geq 0$.

If, in contrast, a system's impulse response has non-zero values for time instants before zero, i.e. $h(t) \neq 0$ for some $t < 0$, this means the system is an "oracle" and produces output in anticipation of an impulse to occur at time zero. Such systems are not physically realisable and are called non-causal systems.



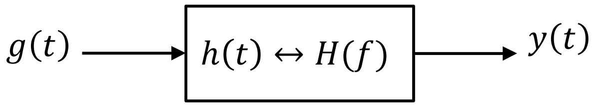
An example to non-causal systems is the ideal LPF. Let us consider the ideal LPF with a cut-off at $f_{cut-off} = 3$ Hz (see, Section 8.A.2). The impulse response of this system can be found using the Fourier Transform synthesis equation (see, Section 9.3), or the inverse Fourier Transform of its impulse response. The resulting impulse response is depicted in the top-right figure. We notice that the output has ripples before the impulse occurs at $t = 0$ and builds up to a peak at $t = 0$ in anticipation of an impulse. Changing the cut-off would only change the period of these ripples and the envelope, but not make the response zero for $t < 0$. As a result, the ideal LPF is a non-causal system.

One can obtain a causal LPF by designing first the impulse response, and then use the Fourier Transform analysis formula to verify its frequency response. We should, however, expect the frequency response to be non-ideal as now we have the constraint on the impulse response to be causal. An example design will use zero for $t < 0$ and a window of the most significant part of the ideal response after $t \geq 0$. Such an impulse response is depicted in the bottom-left figure. The corresponding frequency response is given inside the system block and reveals that the frequency response is now not “ideal”: It has ripples in its amplitude response, and non-zero phase in contrast to the ideal response. Nevertheless, this system is physically realisable and imperfection might be ignored for resulting with small deviations from the ideal output.

9.6 Input/Output Relations of LTI Systems

BONUS

2. **The impulse response** $h(t)$ of an LTI system specifies its output $y(t)$ in response to an input $g(t)$ through an operation known as **convolution**:



$$y(t) = \int_{-\infty}^{\infty} g(\tau)h(t-\tau)d\tau \quad (12)$$

- If the system is causal, then $h(t)$ is zero for $t < 0$, thus the convolution operation becomes

$$y(t) = \int_{-\infty}^{t} g(\tau)h(t-\tau)d\tau \quad (13)$$

The input/output relation of an LTI system can be established based on its response to an impulse $\delta(t)$, i.e. its impulse response $h(t)$.

This relation is given by Equation (12), which is known as the convolution integral (or convolution operation). Equation (12) finds the output of an LTI system in response to the input signal $g(t)$ in terms of its impulse response $h(t)$. In other words, the convolution integral transforms the response to an impulse, to the response to an arbitrary input $g(t)$. If the system is causal, then $h(t)$ is zero for all $t < 0$. Therefore, the integrand is zero for all $\tau > t$ and the upper limit of the integral can be replaced with t without affecting the result. The resulting convolution integral for a causal LTI system is given in equation (13).

For further information on the topic, see Section 2 in [1].

10 Summary

- Real number line, a rotating wheel on a plane, and trigonometric functions
- Complex number plane, the rotating wheel and complex exponentials
- Signals as complex/real functions of time
- Two sides of the same coin #1: Periodic signals and Fourier series coefficients
- Two sides of the same coin #2: Non-periodic signals and frequency-domain representations via The Fourier Transform
- Systems as entities outputting signals in response to input signals.
- Interconnections and basic building blocks
- Linear and Time Invariant Systems (LTI), and their response to complex exponential (harmonic) signals
- The frequency-response of LTI systems
- Characterisation of LTI systems by their impulse response, causality, the convolution operation

The goal of these notes was to introduce the student to the basic concepts in signals (or time functions) and systems (or signal transformers). In particular, time-frequency representations of signals were one of the main points considered. The frequency-domain representation of signals proved useful when investigating how systems transform them and output new signals in response. The second main point was connected to this view of systems and was the frequency response of an important class of systems referred to as linear-time invariant (LTI) systems.

The exposition started with the set of real numbers $\mathbb{R}$ and the real line. Then, we considered a rotating wheel on a plane and the trigonometric functions to find the coordinates of a point on the wheel. We demonstrated how the same problem can be treated using complex numbers on a the complex number plane, and introduced complex exponentials to model a point on a rotating wheel.

Then, we showed that signals as time functions can be decomposed as a weighted sums of complex exponentials: periodic functions can be represented as a series sum of complex exponentials that have integer multiples of the fundamental frequency, and non-periodic functions can be represented as an integral of (infinitesimally) weighted complex exponentials across the entire frequency spectrum. Therefore, time-domain and frequency-domain representations of signals are two sides of the same coin.

Next, we introduced systems as entities that output signals in response to input signals. Specifically, we focused on LTI systems and characterised them with their frequency-response that reveals how the output will weight the complex exponential contents of the input. For the interested student, we extended the discussion to include the time-domain impulse response of an LTI system, and how it characterises the causality/physical realisability of the system. Also introduced was the convolution operation that uniquely specifies the output signal in the time-domain using the input signal and the system impulse response.

References

[1] Alan V. Oppenheim, Alan S. Willsky, S. Hamid Nawab, *Signals and Systems*, 2nd Edition, Prentice Hall, 2010.
[2] Simon Haykin, Michael Moher, *Communication Systems*, 5th Edition, John Wiley & Sons, 2009.

Further Reading

1. Chapter 2 “Representation of Signals and Systems” in [2]
2. Chapter 1 “Signals and Systems” in [1]
3. Chapter 2 “Linear Time Invariant Systems” in [1]
4. Chapter 3 “Fourier Series Representation of Periodic Signals” Sections 3.0-3.5 in [1]

Links to online content on Fourier series/transform (repeated)

1. The exponential function: https://www.youtube.com/watch?v=v0YEaeIClKY
2. Fourier Series: https://www.youtube.com/watch?v=r6sGWTCMz2k
3. Fourier Transform: : https://www.youtube.com/watch?v=spUNpyF58BY

Most of the contents of these notes can be found in further detail in the two references on the slide. Reference [1] provides a complete exposition of signals and systems, so introductory sections should be considered for further reading. A list of specific sections for self-study are listed on the slide. Reference [2] has a single chapter reviewing signals & systems. The students are encouraged to read this Chapter 2 of [2] for further details.

There are some excellent online content on these topics. Three links that were previously cited in these notes are given on this slide. The students are strongly encouraged to watch the visualisations in these videos.